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Abstract

In this work, we study continuous reformulations of zero-one programming problems. We prove that, under suitable conditions, the optimal solutions of a zero-one programming problem can be obtained by solving a specific continuous problem.

Keywords Zero-one programming, Concave functions, Continuous programming

1 Introduction

Several important problems arising in operations research, graph theory and mathematical programming are formulated as 0-1 programming problems. A possible approach for solving this class of problems can be that of transforming the original problem into an equivalent continuous problem. Various transformations have been proposed in the literature (see e.g. [1]-[3], [10]-[12]). A well-known continuous reformulation comes out by relaxing the integrality constraints on the variables and by adding a penalty term to the objective function. This approach has been first introduced by Raghavachari [13] to solve 0-1 linear programming problems. There are many other papers related to the one by Raghavachari (see e.g. [4],[6]-[9], [14] and [16]).
In this paper, we propose a different continuous reformulation for solving 0-1 programming problems obtained by relaxing the integrality constraints on the variables and by making a nonlinear transformation of the variables in the objective function. It can be proved that, under suitable assumptions, a given binary problem and its continuous reformulation are equivalent. The paper is organized as follows. In Section 2, we show a general equivalence result between a 0-1 programming problem and a continuous problem. In Section 3, we define various continuous reformulations, and we show (using the general results stated in Section 2) that a binary problem and its continuous reformulations share the same global minima.

2 Equivalent continuous reformulations for zero-one programming problems

We start from the zero-one programming problem

$$\begin{aligned} &\min c^T x \\ &\text{s.t.} \;\; x \in C \\ &\qquad x \in \{0,1\}^n \end{aligned} \tag{IP}$$

where $C \subset \mathbb{R}^n$ is a convex set.
Then we consider the following nonlinear constrained problem

$$\begin{aligned} &\min f(x) \\ &\text{s.t.} \;\; x \in C \\ &\qquad 0 \leq x \leq e \end{aligned} \tag{CP}$$

where

$$f(x) = \sum_{\substack{i=1 \\ c_i>0}}^{n} c_i \; g_i(x_i) + \sum_{\substack{i=1 \\ c_i<0}}^{n} |c_i| \; g_i(x_i) + \sum_{\substack{i=1 \\ c_i=0}}^{n} g_i(x_i) - \sum_{\substack{i=1 \\ c_i<0}}^{n} |c_i|, \tag{1}$$

and $g_i : [0,1] \to \mathbf{R}$, $i = 1, \ldots, n$ are continuous concave functions.
In order to prove the equivalence between Problem (IP) and Problem (CP), we now make some assumptions on the set of extreme points of (CP) and on the functions g_i used in the definition of f.

Assumption 1 *Let S be the set of extreme points of (CP). Let x_l and x_u be two values defined as follows:*

$$x_l = \inf_{x \in S} \{x_i \, : \, i = 1, \ldots, n; \; x_i \neq 0\},$$

$$x_u = \sup_{x \in S}\{x_i : i = 1, \dots, n;\ x_i \neq 1\}.$$

We assume there exists a value $\epsilon > 0$ such that $x_l > \epsilon$ and $1 - x_u > \epsilon$.

In Fig. 1 we have an example of a convex feasible set of (CP) satisfying Assumption 1.

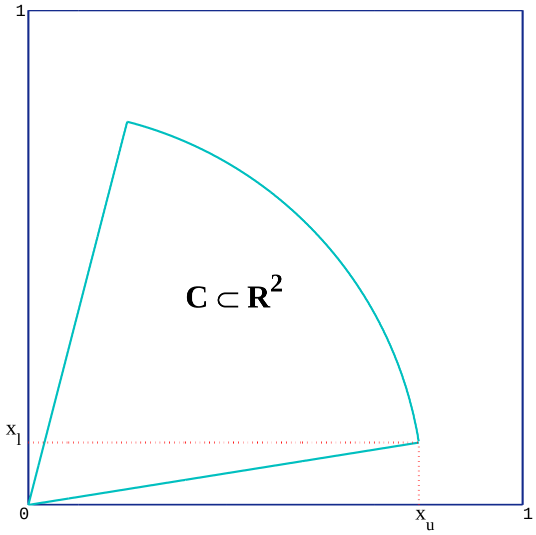

Figure 1: Example of a feasible set C.

Assumption 2 *For all indices i such that $c_i > 0$, we have*

(i) $g_i(0) = 0, \quad g_i(1) = 1$;

(ii) $g_i(x_i) > \frac{(n+1)\max_i |c_i| + \sum_i |c_i|}{\min_i |c_i|}$ *if* $x_i \in [x_l, x_u]$;

For all indices i such that $c_i < 0$, we have

(iii) $g_i(0) = 1, \quad g_i(1) = 0$;

(iv) $g_i(x_i) > \frac{(n+1)\max_i |c_i| + \sum_i |c_i|}{\min_i |c_i|}$ *if* $x_i \in [x_l, x_u]$;

For all indices i such that $c_i = 0$, we have

(v) $g_i(0) = 0, \quad g_i(1) = 0$;

(vi) $g_i(x_i) > n \max_i |c_i| + \sum_i |c_i|$ *if* $x_i \in [x_l, x_u]$.

We report here some important results about the minimization of a concave function over a closed convex set (See [15] for further details):

Proposition 1 *Let f be a concave function, and let C be a closed convex set contained in dom f. Suppose there are no half-lines in C on which f is unbounded below. Then:*

$$inf\ \{f(x) \mid x\ \in\ C\}\ =\ inf\ \{f(x) \mid x\ \in\ E\}, \tag{2}$$

where E is the subset of C consisting of the extreme points of $C \cap L^{\perp}$, being L the lineality space of C and $L^{\perp}$ the orthogonal complement of L. The infimum relative to C is attained only when the infimum relative to E is attained.

The following results are an immediate consequence of Proposition 1:

Corollary 1 *Let f be a concave function, and let C be a closed convex set contained in dom f. Suppose that C contains no lines. Then, if the infimum of f relative to C is attained at all, it is attained at some extreme points of C.*

Corollary 2 *Let f be a concave function, and let C be a nonempty polyhedral convex set contained in dom f. Suppose there are no half-lines in C on which f is unbounded below. Then the infimum of f relative to C is attained.*

Combining Corollary 1 and 2 we obtain the following result:

Corollary 3 *Let f be a concave function, and let C be a nonempty polyhedral convex set contained in dom f. Suppose that C contains no lines, and that f is bounded below on C. Then the infimum of f relative to C is attained at one of the (finitely many) extreme points of C.*

Now we can prove the equivalence between the zero-one programming problem (IP) and its continuous concave reformulation (CP).

Theorem 1 *If Assumptions 1 and 2 hold, then problems (IP) and (CP) have the same minimum points.*

Proof. We first prove that if x^* is a solution of (IP) then x^* is a solution of (CP).
Let x^* be a solution of (IP) and suppose by contradiction that there exists a point $\bar{x}$ solution of (CP) such that

$$f(\bar{x}) < f(x^*) = c^T x^*. \tag{3}$$

We consider two cases:

1. Suppose that $\bar{x}_i \in \{0, 1\}$ for all $i = 1, \ldots, n$:

   $$c^T \bar{x} = f(\bar{x}) < f(x^*) = c^T x^*.$$

   This cannot be the case as it would exists $\bar{x} \in C \cap \{0, 1\}^n$ such that $c^T \bar{x} < c^T x^*$, contradicting the fact that x^* is the optimum of (IP).

2. Suppose now that $\exists\, j \in \{1, \ldots, n\}$ s.t. $\bar{x}_j \notin \{0, 1\}$:

   If $c_j > 0$, by (i) and (ii) in Assumption 2 we have:

   $$f(\bar{x}) = \sum_{i:\bar{x}_i=1} c_i + c_j\ g_j(\bar{x}_j) \geq -\sum_i |c_i| + \min_i |c_i|\ g_j(\bar{x}_j) > n\ \max_i |c_i|.$$

If $c_j < 0$ by *(iii)* and *(iv)* in Assumption 2, we have:

$$f(\bar{x}) = \sum_{i:\bar{x}_i=1} c_i + |c_j|\ g_j(\bar{x}_j) - |c_j| \geq -\sum_i |c_i| + \min_i |c_i|\ g_j(\bar{x}_j) - max|c_j| > n \max_i |c_i|.$$

Finally if $c_j = 0$, by *(v)* and *(vi)* in Assumption 2, we have:

$$f(\bar{x}) = \sum_{i:\bar{x}_i=1} c_i + g_j(\bar{x}_j) \geq -\sum_i |c_i| + g_j(\bar{x}_j) > n \max_i |c_i|.$$

Hence, for each $x \in C \cap \{0,1\}^n$ we have

$$f(\bar{x}) > n \max_i |c_i| > c^T x \tag{4}$$

which implies

$$f(\bar{x}) > c^T x^*,$$

but this contradicts (3).

We now prove that if $\bar{x}$ is a solution of (CP) then $\bar{x}$ is a solution of (IP).

If $\bar{x}$ is a solution of (CP) then, since we are minimizing a concave function over a compact and convex set, by Corollary 1, we have that $\bar{x}$ is an extreme point of $C \cap [0,1]^n$.

We first prove that $\bar{x} \in \{0,1\}^n$.

By contradiction, we suppose that there exists an index $j \in \{1,\dots,n\}$ such that $\bar{x}_j \notin \{0,1\}$. By repeating the same arguments used in the first part of the proof (case 2), for all $x \in C \cap \{0,1\}^n$ we have

$$f(\bar{x}) > c^T x = f(x)$$

thus obtaining a contradiction.

Now suppose by contradiction that there exists x^*, solution of (IP), such that

$$c^T x^* < c^T \bar{x}. \tag{5}$$

Since $\bar{x} \in \{0,1\}^n$ we have that $f(\bar{x}) = c^T \bar{x}$, thus (5) implies that $f(x^*) < f(\bar{x})$, contradicting the optimality of $\bar{x}$ for (CP). The Theorem is then proved. □

We can apply the previous result to the case of zero-one linear programing problems. Suppose that we are dealing with the following problem:

$$\begin{aligned} &\min c^T x \\ &\text{s.t. } x \in P \\ &\quad\ \ x \in \{0,1\}^n \end{aligned} \tag{ILP}$$

where P is a polyhedral set. Then we can prove the equivalence of (ILP) with the following problem

$$\begin{aligned} &\min f(x) \\ &\text{s.t. } x \in P \\ &\quad\ \ 0 \leq x \leq e \end{aligned} \tag{LP}$$

where the function $f : [0,1]^n \to \mathbf{R}$ is defined as in (1). The following result is a straightforward application of Theorem 1.

Proposition 2 *If Assumption 2 holds, then problems (ILP) and (LP) have the same minimum points.*

Proof. First of all, we see that the feasible set of problem (LP) satisfies Assumption 1. Let V be the set of the vertices of the polyhedron P. Since the cardinality of V is finite, we can define

$$x_l = \min_{x \in V} \{x_i : i = 1, \ldots, n;\ x_i \neq 0\},$$

$$x_u = \max_{x \in V} \{x_i : i = 1, \ldots, n;\ x_i \neq 1\}.$$

It is easy to see that there exists a value $\epsilon > 0$ such that $x_l > \epsilon$ and $1 - x_u > \epsilon$. In other words, x_l and x_u are respectively the minimum and the maximum component (different from 0 and 1) of the vertices of the polyhedron P.
The rest of the proof is a verbatim repetition of Theorem 1. □

3 Examples of continuous reformulations

In this section, starting from the ideas developed in [5], we propose various examples of continuous reformulations for solving a given zero-one programming problem, and we show (using the general results stated in the previous section) that these reformulations have the same global minimizers of the original zero-one programming problem.
First of all, we denote

$$\tilde{c} = \frac{(n+1)\max_i |c_i| + \sum_i |c_i|}{\min_i |c_i|}\ .$$

Now we can define the functions g_i to be used in (1):

Exponential Functions

Case $c_i > 0$:

$$g_i(t) = \min\left\{\gamma_{1_+}(1 - e^{-\alpha t}), 1 + \gamma_{2_+}(1 - e^{-\alpha(1-t)})\right\}, \tag{6}$$

$$\gamma_{1_+} > \frac{\tilde{c}}{1 - e^{-\alpha x_l}}\,, \qquad \gamma_{2_+} > \frac{\tilde{c} - 1}{1 - e^{-\alpha(1-x_u)}}; \tag{7}$$

Case $c_i < 0$:

$$g_i(t) = \min\left\{1 + \gamma_{1_-}(1 - e^{-\alpha t}), \gamma_{2_-}(1 - e^{-\alpha(1-t)})\right\}, \tag{8}$$

$$\gamma_{1_-} > \frac{\tilde{c} - 1}{1 - e^{-\alpha x_l}}\,, \qquad \gamma_{2_-} > \frac{\tilde{c}}{1 - e^{-\alpha(1-x_u)}}; \tag{9}$$

Case $c_i = 0$:

$$g_i(t) = \min\Big\{\gamma_0(1-e^{-\alpha t}), \gamma_0(1-e^{-\alpha(1-t)})\Big\}, \tag{10}$$

$$\gamma_0 > \max\Big\{\frac{n\max_i |c_i| + \sum_i |c_i|}{1-e^{-\alpha x_l}}, \frac{n\max_i |c_i| + \sum_i |c_i|}{1-e^{-\alpha(1-x_u)}}\Big\}; \tag{11}$$

with $\alpha > 0$.

Logistic functions

Case $c_i > 0$:

$$g_i(t) = \min\left\{\delta_{1_+}\left(\frac{1-e^{-\alpha t}}{2(1+e^{-\alpha t})}\right), 1 + \delta_{2_+}\left(\frac{1-e^{-\alpha(1-t)}}{2(1+e^{-\alpha(1-t)})}\right)\right\}, \tag{12}$$

$$\delta_{1_+} > \tilde{c}\frac{2(1+e^{-\alpha x_l})}{1-e^{-\alpha x_l}}, \qquad \delta_{2_+} > (\tilde{c}-1)\frac{2(1+e^{-\alpha(1-x_u)})}{1-e^{-\alpha(1-x_u)}}; \tag{13}$$

Case $c_i < 0$:

$$g_i(t) = \min\left\{1 + \delta_{1_-}\left(\frac{1-e^{-\alpha t}}{2(1+e^{-\alpha t})}\right), \delta_{2_-}\left(\frac{1-e^{-\alpha(1-t)}}{2(1+e^{-\alpha(1-t)})}\right)\right\}, \tag{14}$$

$$\delta_{1_-} > (\tilde{c}-1)\frac{2(1+e^{-\alpha x_l})}{1-e^{-\alpha x_l}}, \qquad \delta_{2_-} > \tilde{c}\frac{2(1+e^{-\alpha(1-x_u)})}{1-e^{-\alpha(1-x_u)}} \tag{15}$$

Case $c_i = 0$:

$$g_i(t) = \min\left\{\delta_0\left(\frac{1-e^{-\alpha t}}{2(1+e^{-\alpha t})}\right), \delta_0\left(\frac{1-e^{-\alpha(1-t)}}{2(1+e^{-\alpha(1-t)})}\right)\right\}, \tag{16}$$

$$\delta_0 > \Big(n\max_i |c_i| + \sum_i |c_i|\Big) \cdot \max\Big\{\frac{2(1+e^{-\alpha x_l})}{1-e^{-\alpha x_l}}, \frac{2(1+e^{-\alpha(1-x_u)})}{1-e^{-\alpha(1-x_u)}}\Big\}; \tag{17}$$

with $\alpha > 0$.

Logarithmic functions

Case $c_i > 0$:

$$g_i(t) = \min\Big\{\alpha_{1_+}(\ln(t+\epsilon) - \ln\epsilon), 1 + \alpha_{2_+}(\ln(1-t+\epsilon) - \ln\epsilon)\Big\}, \tag{18}$$

$$\alpha_{1_+} > \frac{\tilde{c}}{\ln(x_l+\epsilon) - \ln\epsilon}, \qquad \alpha_{2_+} > \frac{\tilde{c}-1}{\ln(1-x_u+\epsilon) - \ln\epsilon}; \tag{19}$$

Case $c_i < 0$:

$$g_i(t) = \min\Big\{1 + \alpha_{1_-}(\ln(t+\epsilon) - \ln\epsilon), \alpha_{2_-}(\ln(1-t+\epsilon) - \ln\epsilon)\Big\}, \tag{20}$$

$$\alpha_{1_-} > \frac{\tilde{c}-1}{\ln(x_l+\epsilon) - \ln\epsilon}, \qquad \alpha_{2_-} > \frac{\tilde{c}}{\ln(1-x_u+\epsilon) - \ln\epsilon}; \tag{21}$$

Case $c_i = 0$:

$$g_i(t) = \min\left\{\alpha_0(\ln(t+\epsilon) - \ln\epsilon), \alpha_0(\ln(1-t+\epsilon) - \ln\epsilon)\right\}, \tag{22}$$

$$\alpha_0 > \max\left\{\frac{n\max_i |c_i| + \sum_i |c_i|}{\ln(x_l+\epsilon) - \ln\epsilon}, \frac{n\max_i |c_i| + \sum_i |c_i|}{\ln(1-x_u+\epsilon) - \ln\epsilon}\right\}; \tag{23}$$

with $\epsilon > 0$.

Hyperbolic functions

Case $c_i > 0$:

$$g_i(t) = \min\left\{\beta_{1_+}(-(t+\epsilon)^{-p} + \epsilon^{-p}), 1 + \beta_{2_+}(-(1-t+\epsilon)^{-p} + \epsilon^{-p})\right\}, \tag{24}$$

$$\beta_{1_+} > \frac{\tilde{c}}{-(x_l+\epsilon)^{-p} + \epsilon^{-p}},, \qquad \beta_{2_+} > \frac{\tilde{c} - 1}{-(1-x_u+\epsilon)^{-p} + \epsilon^{-p}}; \tag{25}$$

Case $c_i < 0$:

$$g_i(t) = \min\left\{1 + \beta_{1_-}(-(t+\epsilon)^{-p} + \epsilon^{-p}), \beta_{2_-}(-(1-t+\epsilon)^{-p} + \epsilon^{-p})\right\}, \tag{26}$$

$$\beta_{1_-} > \frac{\tilde{c} - 1}{-(x_l+\epsilon)^{-p} + \epsilon^{-p}}, \qquad \beta_{2_-} > \frac{\tilde{c}}{-(1-x_u+\epsilon)^{-p} + \epsilon^{-p}}; \tag{27}$$

Case $c_i = 0$:

$$g_i(t) = \min\left\{\beta_0(-(t+\epsilon)^{-p} + \epsilon^{-p}), \beta_0(-(1-t+\epsilon)^{-p} + \epsilon^{-p})\right\}, \tag{28}$$

$$\beta_0 > \max\left\{\frac{n\max_i |c_i| + \sum_i |c_i|}{-(x_l+\epsilon)^{-p} + \epsilon^{-p}}, \frac{n\max_i |c_i| + \sum_i |c_i|}{-(1-x_u+\epsilon)^{-p} + \epsilon^{-p}}\right\}; \tag{29}$$

with $\epsilon > 0$.

In Fig. 2, we report the various functions that can be used in the reformulation of a zero-one programming problem. By setting the functions g_i equal to the exponential terms, we can define the objective function of the continuous problem (CP) as follows:

$$\begin{aligned} f(x) = \quad & \sum_{\substack{i=1 \\ c_i>0}}^{n} c_i \min\left\{\gamma_{1_+}(1 - e^{-\alpha x_i}), 1 + \gamma_{2_+}(1 - e^{-\alpha(1-x_i)})\right\} \\ & + \sum_{\substack{i=1 \\ c_i<0}}^{n} |c_i| \min\left\{1 + \gamma_{1_-}(1 - e^{-\alpha x_i}), \gamma_{2_-}(1 - e^{-\alpha(1-x_i)})\right\} \\ & + \sum_{\substack{i=1 \\ c_i=0}}^{n} \min\left\{\gamma_0(1 - e^{-\alpha x_i}), \gamma_0(1 - e^{-\alpha(1-x_i)})\right\} - \sum_{\substack{i=1 \\ c_i<0}}^{n} |c_i|. \end{aligned} \tag{30}$$

Now we can prove that for a particular choice of the functions g_i, problems (IP) and (CP) are equivalent.

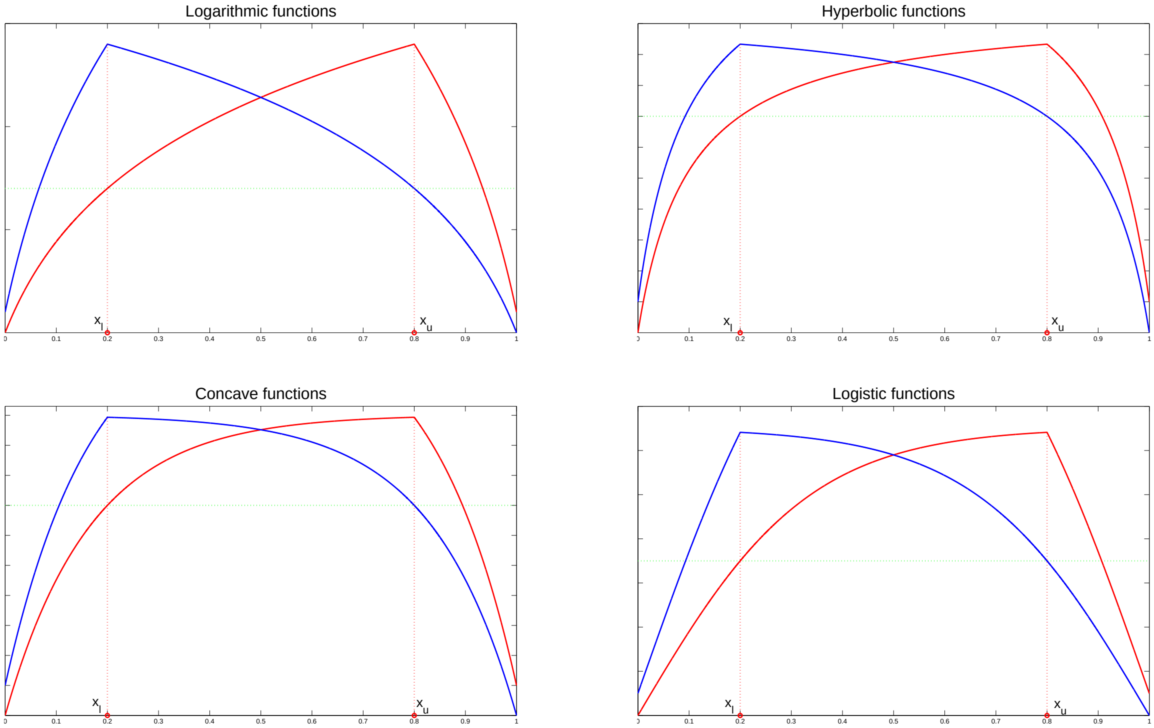

Figure 2: Examples of functions g_i for $c_i > 0$ (in red) and $c_i < 0$ (in blue). The green line represents $y(x) = \tilde{c}$.

Proposition 3 *If Assumption 1 holds and f is defined as in (30), then problems (IP) and (CP) have the same minimum points.*

Proof. We only need to prove that the functions g_i used in the definition of the objective function (30), satisfy Assumption 2.
Since $\tilde{c} > 1$ and

$$\epsilon < x_l \leq x_u < 1 - \epsilon,$$

we have that all the γ-parameters are strictly greater than zero.
We consider three different cases:

1. $c_i > 0$: the choice of the parameters γ_{1_+} and γ_{2_+}, and the fact that $\tilde{c} > 1$ guarantee,

$$g(0) = \min\left\{0, 1 + \gamma_{2_+}(1 - e^{-\alpha})\right\} = 0$$

and

$$g(1) = \min\left\{\gamma_{1_+}(1 - e^{-\alpha}), 1\right\} = 1.$$

Furthermore, for all $x_i \in [x_l, x_u]$, we have

$$g_i(x_i) = \min\left\{\gamma_{1_+}(1 - e^{-\alpha x_i}), 1 + \gamma_{2_+}(1 - e^{-\alpha(1-x_i)})\right\} >$$

$$> \min\left\{\frac{\tilde{c} \cdot (1 - e^{-\alpha x_i})}{1 - e^{-\alpha x_l}}, 1 + \frac{(\tilde{c} - 1) \cdot (1 - e^{-\alpha(1-x_i)})}{1 - e^{-\alpha(1-x_u)}}\right\} > \tilde{c}.$$

2. $c_i < 0$: the choice of the parameters γ_{1_-} and γ_{2_-}, and the fact that $\tilde{c} > 1$ guarantee,

$$g_i(0) = \min\left\{1, \gamma_{2_-}(1 - e^{-\alpha})\right\} = 1$$

and

$$g_i(1) = \min\left\{1 + \gamma_{1_-}(1 - e^{-\alpha}), 0\right\} = 0.$$

Furthermore, for all $x_i \in [x_l, x_u]$, we have

$$g_i(x_i) = \min\left\{1 + \gamma_{1_-}(1 - e^{-\alpha x_i}), \gamma_{2_-}(1 - e^{-\alpha(1-x_i)})\right\} >$$
$$> \min\left\{1 + \frac{(\tilde{c} - 1) \cdot (1 - e^{-\alpha x_i})}{1 - e^{-\alpha x_l}}, \frac{\tilde{c} \cdot (1 - e^{-\alpha(1-x_i)})}{1 - e^{-\alpha(1-x_u)}}\right\} > \tilde{c}.$$

3. $c_i = 0$: the choice of the parameter γ_0 guarantees

$$g_i(0) = \min\left\{0, \gamma_0(1 - e^{-\alpha})\right\} = \min\left\{\gamma_0(1 - e^{-\alpha}), 0\right\} = g_i(1) = 0.$$

Furthermore, for all $x_i \in [x_l, x_u]$, we have

$$g_i(x_i) = \min\left\{\gamma_0(1 - e^{-\alpha x_i}), \gamma_0(1 - e^{-\alpha(1-x_i)})\right\} > n \max_i |c_i| + \sum_i |c_i|. \quad (31)$$

Then Assumption 2 is satisfied.

□

The following result is obtained as an immediate consequence of Proposition 2:

Corollary 4 *If function f is defined as in (30), then problems (ILP) and (LP) have the same minimum points.*

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