Annali del Dipartimento di Metodi e Modelli per l'Economia il Territorio e la Finanza
2022, DOI: 10.13133/2611-6634/1392

E-ISSN 2611-6634

ISSN 2385-0825

Research article

First published online: December 10, 2022

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PRICING THROUGH THE CHOQUET INTEGRAL

Abstract

The classical no-arbitrage pricing theory allows to price assets through a linear pricing rule, by assuming a frictionless and competitive market. Moreover, completeness of the market assures that the pricing rule is defined as a discounted expected value with respect to a unique equivalent martingale measure. On the other hand, under no-arbitrage assumption, incomplete models, such as the trinomial model, lead to a set of equivalent martingale measures. This suggests to work with non-linear pricing rules that can allow frictions in the market. A generalized pricing rule can be achieved by replacing additive measures with non-additive measures such as convex capacities and belief functions in Dempster-Shafer theory. The paper recaps results on non-additive measures and Choquet expectation as non-linear functional to be used in pricing. In the literature it has been proved that, under suitable conditions, a non-linear pricing rule can be expressed as a Choquet expectation with respect to a convex capacity. In the trinomial market model the lower probability is a belief function, but it cannot be used to reach the lower expectation through the Choquet integral. Nevertheless it can avoid a generalized Dutch book condition in the framework of partially resolving uncertainty.

Keywords: incomplete markets, non-linear pricing rule, Choquet integral, belief functions, generalized no-Dutch book.

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1 Introduction

The classical pricing theory is based on the assumptions that the market is frictionless and competitive. Hence, the existence of a linear pricing rule is equivalent to the fact that the market is arbitrage-free (the first fundamental theorem of asset pricing). In turn, the existence of a linear pricing rule is equivalent to the existence of an equivalent martingale measure. Moreover, the assumption of *completeness* of the market assures that the equivalent martingale measure is unique.

Under market incompleteness, the uniqueness is lost and this leads to a set of equivalent martingale measures (see, e.g., Amihud and Mendelson, 1986; Chen and Kulperger, 2006; Acciaio et al., 2016). The literature concerning the theory of sets of probability measures and their envelopes essentially refers to Walley (1991), Gilboa and Schmeidler (1989), Schmeidler (1989), Cozman (2000), Ghirardato and Marinacci (2001), Capotorti et al. (2008), Coletti et al. (2016), Erreygers et al. (2019), Petturiti and Vantaggi (2020), T'Joens et al. (2021), Petturiti and Vantaggi (2022).

In the framework of decision theory, sets of probability measures are related also to the notion of ambiguity (Etner et al., 2012; Gilboa and Marinacci, 2011).

As is well-known, the simplest example of incomplete market is the *trinomial market model*. In the classical approach the market can be completed by adding another risky asset that leads to choose a specific equivalent martingale measure in the original set. Anyhow the latter procedure requires a choice criterion and it would lead to lose some information contained in the set. More generally, incompleteness continues to hold if the risky asset is allowed to have n different possible future values, for $n \geq 3$.

The existence of a set of probability measures suggests to work with a non-linear pricing rule that can model frictions in the market. Frictions such as bid-ask spreads are largely proved to exist (Amihud and Mendelson, 1986, 1991) and they are studied in Bensaid et al. (1992), Jouini and Kallal (1995), Acciaio et al. (2016), Cerraia-Vioglio et al. (2015), Chateauneuf et al. (1996), Chateauneuf and Cornet (2022).

There are alternative attempts along this line by considering different functionals for pricing: envelopes of expected values with respect to a class of probability measures, integral forms such as Choquet expectation with respect to *non-additive measures*. In general, the two approaches are not equivalent but in case of a convex capacity (or a belief function) ν the Choquet integral coincides with the lower expectation induced by its core (see Schmeidler, 1986). In particular, in Cinfrignini et al. (2021) the study of market frictions has been faced by replacing probability measures with belief functions in the Dempster-Shafer theory (Dempster, 1967; Shafer, 1976).

The paper is structured as follows. In Section 2 we report the classical no-arbitrage pricing theory in the one-period setting. We introduce complete and incomplete markets and we show the one-period trinomial market model as a prototypical example of incomplete market. Section 3 introduces non-additive measures that are required to deal with non-linear pricing rules and the Choquet integral as non-linear functional. In Section 4 we recall and connect some results given in Chateauneuf et al. (1996) and Coletti et al. (2020) assuring that a non-linear pricing rule can be expressed through a Choquet expectation. In particular we will focus on a *global* lower pricing rule that can be expressed as a discounted Choquet expectation with respect to a convex capacity or a belief function. Then we point out that in the trinomial market model the lower probability, proved to be a belief function in Cinfrignini et al. (2021), gives rise to a Choquet expectation that does not coincide with the lower expectation induced by the equivalent martingale measures. Nevertheless, the lower price assessment on the bond and the risky asset satisfies the generalized no-Dutch-book condition obtained from Coletti et al. (2020). Finally, the last section draws conclusions.

2 Classical one-period no-arbitrage theory

We refer to a one-period financial market open at times $t = 0$ and $t = 1$. An *asset* (or security) is a tradable financial instrument that has a positive or negative cash flow of money. The cash flow is deterministic (i.e. it does not depend on future states of the world) when the asset is *riskless*; otherwise the cash flow is a random variable since it depends on what state of world will occur, and the asset is called *risky*. The market is based on two fundamental assumptions (Allingham, 1991):

(i) absence of frictions (there are no transaction costs, taxes and others restrictions on trading);

(ii) competitiveness (every quantity can be traded at market's price).

One period market model consists of a set of K risky assets with price process $(S_0^{(k)}, S_1^{(k)})$, for $k = 1, \dots, K$, and by one riskless asset (bond) with price process (B_0, B_1) that is identified with a 0-th asset $(S_0^{(0)}, S_1^{(0)})$ to simplify the notation. It is usually assumed that $S_0^{(k)} = s^{(k)} > 0$ is a deterministic positive value (called *price*), while $S_1^{(k)}$ is a random variable (called *payoff*), for each $k = 1, \dots, K$. The bond process, without loss of generality, is assumed to be $S_0^{(0)} = 1$ and $S_1^{(0)} = 1 + r$, where $r > 0$ is the risk-free interest rate of the market.

Price processes are defined on a filtered probability space $(\Omega, \{\mathcal{F}_0, \mathcal{F}_1\}, \mathcal{F}, P)$ where $\Omega = \{1, \dots, n\}$, $n \in \mathbb{N}$ is a finite state space, $\{\mathcal{F}_0, \mathcal{F}_1\}$ is a filtration such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_1 = \mathcal{F} = \mathcal{P}(\Omega)$ is the power set of Ω, and P is a probability measure on $\mathcal{F}$. The probability measure is called *"natural"* or *"real-world"* probability measure and the classical pricing theory asks for the positivity of P since it assures that an asset with a non-negative and non-null payoff will have a positive price at time $t = 0$. We also denote by $\mathbb{R}^\Omega$ the set of all random variables which are automatically $\mathcal{F}$-measurable. Moreover, scalar real numbers are identified with constant random variables. Finally, $\mathbf{P}(\Omega, \mathcal{F})$ stands for the set of all probability measures on $(\Omega, \mathcal{F})$.

Let us denote the set of all random payoff with $\mathcal{G} = \{S_1^{(0)}, \dots, S_1^{(K)}\}$ and with $\pi : \mathcal{G} \to \mathbb{R}$ a function such that $\pi\left(S_1^{(k)}\right) = S_0^{(k)}$, for $k = 0, \dots, K$, which is called *price assessment*. Our aim is to look for a global *pricing rule* $\pi' : \mathbb{R}^\Omega \to \mathbb{R}$ that extends π.

The risk-free bond is usually used as a numéraire (see Pliska, 1997); it means that the riskless bond allows to discount the risky process and defines a new process denoted as $\left(\tilde{S}_0^{(k)}, \tilde{S}_1^{(k)}\right)$ with $\tilde{S}_0^{(k)} = S_0^{(k)}$ and $\tilde{S}_1^{(k)} = (1+r)^{-1} S_1^{(k)}$, for $k = 1, \dots, K$.

A *portfolio* (or *trading strategy*) is a collection of assets that an agent can hold. It is denoted by a vector $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_K) \in \mathbb{R}^{K+1}$, whose component λ_k expresses the number of units purchased ($\lambda_k > 0$) or sold ($\lambda_k < 0$) of the k-th asset in the time interval $[0, 1]$.

The *price* at time $t = 0$ of the portfolio $\boldsymbol{\lambda}$ is computed as weighted sum of prices:

$$V_0^{\boldsymbol{\lambda}} = \sum_{k=0}^{K} \lambda_k S_0^{(k)} = \sum_{k=0}^{K} \lambda_k \pi\left(S_1^{(k)}\right); \tag{1}$$

while the *payoff* of the portfolio $\boldsymbol{\lambda}$ is given by a random variable $V_1^{\boldsymbol{\lambda}} : \Omega \to \mathbb{R}$ defined, for every $i \in \Omega$, as the weighted sum of payoffs:

$$V_1^{\boldsymbol{\lambda}}(i) = \sum_{k=0}^{K} \lambda_k S_1^{(k)}(i). \tag{2}$$

Given the set of random variables $\mathcal{G}$, $\boldsymbol{\lambda} \in \mathbb{R}^{K+1}$ is a *Dutch-book* portfolio if the following condition holds:

$$\max_{i \in \Omega} \sum_{k=0}^{K} \lambda_k \left(\tilde{S}_1^{(k)}(i) - \pi\left(S_1^{(k)}\right)\right) < 0. \tag{3}$$

The condition means that the portfolio $\boldsymbol{\lambda} \in \mathbb{R}^{K+1}$ gives rise to a *sure loss* for each $i \in \Omega$, since the supremum gain is negative for sure. The portfolio is also called *incoherent*. Conversely, if inequality in Equation (3) does not hold, the portfolio is called *coherent* and it avoids a *Dutch-book* opportunity, i.e. it avoids a sure loss (Schervish et al., 2008).

The arbitrage definition is stronger than that of Dutch-book, since the former guarantees a positive payoff in, at least, one state of the world, with a zero or negative price. A portfolio $\boldsymbol{\lambda} \in \mathbb{R}^{K+1}$ is an arbitrage portfolio if one of the following condition holds (Allingham, 1991):

(1) $V_0^{\boldsymbol{\lambda}} \leq 0$ and $V_1^{\boldsymbol{\lambda}} \geq 0$ with a strict inequality for at least one $i \in \Omega$;

(2) $V_0^{\boldsymbol{\lambda}} < 0$ and $V_1^{\boldsymbol{\lambda}} = 0$.

Equivalently $\boldsymbol{\lambda} \in \mathbb{R}^{K+1}$ is an arbitrage portfolio if $\sum_{k=0}^{K} \lambda_k \left(\tilde{S}_1^{(k)}(i) - \pi\left(S_1^{(k)}\right)\right) \geq 0$, for all i, with a strict inequality for at least one $i \in \Omega$. Note that a Dutch-book opportunity implies the existence of an arbitrage but the converse does not hold (Schervish et al., 2008).

The assumption that the market has to be *arbitrage-free* is standard in classical pricing theory (see, e.g., Pliska, 1997; Dybvig and Ross, 1989) and it has important implications in asset pricing. The absence of arbitrage opportunities guarantees the existence of a positive linear pricing rule $\pi' : \mathbb{R}^{\Omega} \to \mathbb{R}$ such that $\pi'\left(S_1^{(k)}\right) = \pi\left(S_1^{(k)}\right)$, for $k = 0, \ldots, K$ (Dybvig and Ross, 1989).

Furthermore, when the market is *complete*, there is a unique linear pricing rule π' given by the discounted expected value computed with respect to a unique risk-neutral probability measure that has to be equivalent to the natural one.

Under completeness, a *derivative* X, that is a financial contract defined as a random process (X_0, X_1) on the filtered probability space $(\Omega, \{\mathcal{F}_0, \mathcal{F}_1\}, \mathcal{F}, P)$, adapted to the filtration $\{\mathcal{F}_0, \mathcal{F}_1\}$, can be perfectly replicated by setting up a replicating strategy $\boldsymbol{\lambda} \in \mathbb{R}^{K+1}$ composed by the risky assets and the bond, such that they have the same final payoff $X_1 = V_1^{\boldsymbol{\lambda}}$.

Then, by the *law of one price*, they have the same price at time $t = 0$:

$$X_0 = V_0^{\boldsymbol{\lambda}}, \tag{4}$$

and its value is computed as discounted expected value of its payoff:

$$X_0 = (1 + r)^{-1}\mathbb{E}_Q(X_1), \tag{5}$$

where Q is the unique equivalent martingale measure. Therefore, we have that $\pi'(\cdot) = (1 + r)^{-1}\mathbb{E}_Q(\cdot)$.

On the other hand, in the case of an incomplete market, the price assessment is consistent with the no-arbitrage assumption but not each derivative in the market can be replicated by a strategy. This leads to a set of equivalent martingale measures $\mathcal{Q}$ such that each $Q \in \mathcal{Q}$ defines a different price.

Given a *non-replicable* derivative with payoff $Y_1 \in \mathbb{R}^{\Omega}$, its fair price can be computed as an interval defined through the closest replicable derivative. If X_1 is the closest replicable derivative of Y_1, the following quantities can be computed:

$$\overline{V}(Y_1) = \inf_{\substack{X_1 \leq Y_1, \\ X_1 \text{ is replicable}}} (1 + r)^{-1}\mathbb{E}_Q(X_1), \quad \underline{V}(Y_1) = \sup_{\substack{X_1 \leq Y_1, \\ X_1 \text{ is replicable}}} (1 + r)^{-1}\mathbb{E}_Q(X_1). \tag{6}$$

The fair price of the derivative has to be in the interval $\left(\underline{V}(Y_1), \overline{V}(Y_1)\right)$, otherwise it gives rise to an arbitrage opportunity (Pliska, 1997). Another approach to select a replicating strategy for a non-replicable derivative is to choose the best replicating strategy among the imperfect strategies through approximations/algorithms (see Cerný, 2009; Bertsimas et al., 2001). Although they are not detailed here, some criteria to choose a replicating strategy can be the following:

(a) ***sub(super)-hedging.*** We look for a strategy $\boldsymbol{\lambda}_S \in \mathbb{R}^{K+1}$ such that $V_1^{\boldsymbol{\lambda}_S} \leq (\geq) Y_1$. Hence the sub-hedging $\underline{V}_0^{\boldsymbol{\lambda}_S}$ and the super-hedging prices $\overline{V}_0^{\boldsymbol{\lambda}_S}$ are the no-arbitrage bounds for the non-replicable payoff Y_1;

(b) ***quadratic risk minimization.*** We look for a strategy $\boldsymbol{\lambda}_{QR} \in \mathbb{R}^{K+1}$ that minimizes the expected value of the quadratic distance between the payoff of the derivative and the value of the portfolio. The following optimization problem has to be solved:

$$\min_{\boldsymbol{\lambda}_{QR}} \mathbb{E}\left[\left(Y_1 - V_1^{\boldsymbol{\lambda}_{QR}}\right)^2\right]; \tag{7}$$

(c) ***shortfall risk minimization.*** We look for a strategy $\boldsymbol{\lambda}_{SR} \in \mathbb{R}^{K+1}$ that minimizes the shortfall risk. It penalizes only deviations in defect but it is less mathematically tractable. The following problem has to be solved:

$$\min_{\boldsymbol{\lambda}_{SR}} \mathbb{E}\left[\left(Y_1 - V_1^{\boldsymbol{\lambda}_{SR}}\right)^+\right]. \tag{8}$$

Another approach to overcome market's incompleteness is to *complete* the market with an appropriate number of extra assets. Let us introduce the matrix notation to go deep into the problem. Payoffs of riskless and risky assets are defined in the matrix $A \in \mathbb{R}^{n \times (K+1)}$:

$$A = \begin{bmatrix} S_1^{(0)}(1) & S_1^{(1)}(1) & \dots & S_1^{(K)}(1) \\ \vdots & \vdots & & \vdots \\ S_1^{(0)}(n) & S_1^{(1)}(n) & \dots & S_1^{(K)}(n) \end{bmatrix}, \tag{9}$$

and the vector of payoff of the derivative is denoted by $\boldsymbol{X} = (X_1(1), \dots, X_1(n)) \in \mathbb{R}^n$. Hence, an arbitrage-free market is complete if and only if the following linear problem has a unique solution:

$$A\boldsymbol{\lambda}^T = \boldsymbol{X}, \tag{10}$$

where $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_k) \in \mathbb{R}^{K+1}$ is the portfolio such that λ_0 is referred to units of risk-free asset $S^{(0)}$ and λ_k is referred to units of risky asset $S^{(k)}$, for $k = 1, \dots, K$. Problem (10) has a unique solution, assumed that there are no *redundant* assets[1], if and only if $rank(A) = n = K + 1$, hence A has to be a square matrix. Otherwise, the following possibilities can occur (Cerný, 2009):

(I1) $rank(A) = n < (K + 1)$: the market is *complete* but there are $K + 1 - n$ redundant assets that lead to $K + 1 - n$ free parameters referred to redundant assets;

(I2) $rank(A) = (K + 1) < n$: the market is *incomplete* since $n - (K + 1)$ assets are lacking. It can be completed by adding the missing number of assets;

(I3) $rank(A) < n, rank(A) < (K+1)$: the market is *incomplete* and there are $(K+1) - rank(A)$ redundant assets.

However, sometimes the completion is not possible or not desirable as it changes the market structure. Also other procedures that introduce additional requirements such as agents' preferences may be not desirable as they change the framework. A way to define a unique price consistent with the no-arbitrage principle without changing the market structure is to compute prices with every $Q \in \mathcal{Q}$ through Equation (5) and define a set of prices $\mathcal{Y}$. Then we could choose the price $Y \in \mathcal{Y}$

1 An asset whose payoff can be written as a linear combination of others assets' payoffs is called redundant since it does not add anything new to the market. If there are no redundant assets, the market's asset are said to be *linearly independent*.

that departs as little as possible from the actually observed in the market. For this procedure we can refer, for example, to Pascucci and Runggaldier (2011).

The simplest example of complete market model is the one-period binomial model (Cox et al., 1979), while an example of incomplete market model is the trinomial model.

The trinomial model is composed by a bond with price process $(B_0 = 1, B_1 = (1+r)B_0)$ and by a risky asset with the following price process:

$$S_0 = s > 0, \quad S_1 = \begin{cases} uS_0 & \text{with probability } p_1, \\ mS_0 & \text{with probability } p_2, \\ dS_0 & \text{with probability } p_3, \end{cases} \tag{11}$$

where $u > m > d > 0$ are parameters, $p_i \in (0,1)$ for $i = 1,2,3$, and $\sum_{i=1}^{3} p_i = 1$. Such model is free of arbitrage if and only if $u > (1+r) > d$ as the binomial model but it is not complete since it occurs condition (I2).

In the trinomial case there is a set of equivalent martingale measures denoted as:

$$\mathcal{Q} = \{Q \in \mathbf{P}(\Omega, \mathcal{F}) : (1+r)^{-1}\mathbb{E}_Q(S_1) = S_0, \ \ Q \sim P\}. \tag{12}$$

The set $\mathcal{Q}$ is a convex set that can be characterized by its extreme points (Runggaldier, 2006) (in particular it is a segment since there are two extreme points):

$$Q^1 = (q_1^1, q_2^1, q_3^1) \quad = \quad \begin{cases} \left(0, \frac{(1+r)-d}{m-d}, \frac{m-(1+r)}{m-d}\right) & \text{if } m \geq (1+r), \\ \left(\frac{(1+r)-m}{u-m}, \frac{u-(1+r)}{u-m}, 0\right) & \text{if } m < (1+r), \end{cases} \tag{13}$$

$$Q^2 = (q_1^2, q_2^2, q_3^2) \quad = \quad \left(\frac{(1+r)-d}{u-d}, 0, \frac{u-(1+r)}{u-d}\right). \tag{14}$$

We stress that extreme points Q^1 and Q^2 are not equivalent to P since they are not positive on $\mathcal{F}$; hence equivalent martingale measures are given by the strict convex combinations of Q^1 and Q^2:

$$\mathcal{Q} = \{Q^\alpha : Q^\alpha = \alpha Q^1 + (1-\alpha)Q^2, \ \ \alpha \in (0,1)\}, \tag{15}$$

with $Q^\alpha \sim P$, for each $Q^\alpha \in \mathcal{Q}$.

At this point, a suitable criterion to choose one measure in the set is required. In the following example we show that each $Q^\alpha \in \mathcal{Q}$ is an equivalent martingale measure consistent with no-arbitrage assumption but it leads to varied prices for the derivative, through Equation (5).

Example 2.1 *Let $S_0 = 100, u = 2, m = \frac{6}{5}, d = \frac{2}{5}$ and, without loss of generality, $r = 0$. Extreme points of the set $\mathcal{Q}$, computed with Equations* (13) (14)*, are:*

$$Q^1 = \left(0, \frac{3}{4}, \frac{1}{4}\right), \quad Q^2 = \left(\frac{3}{8}, 0, \frac{5}{8}\right).$$

Then the set of equivalent martingale measures is given by:

$$\mathcal{Q} = \left\{Q^\alpha : Q^\alpha = \alpha\left(0, \frac{3}{4}, \frac{1}{4}\right) + (1-\alpha)\left(\frac{3}{8}, 0, \frac{5}{8}\right), \ \ \alpha \in (0,1)\right\}.$$

For instance, let $\alpha = 0.2$. The equivalent martingale measure is $Q^{0.2} = \left(\frac{6}{20}, \frac{3}{20}, \frac{11}{20}\right)$ and we can verify that $Q^{0.2} \in \mathcal{Q}$ by computing the following expected value:

$$\mathbb{E}_{Q^{0.2}}\left(\frac{S_1}{S_0}\right) = 2 \cdot \frac{6}{20} + \frac{6}{5} \cdot \frac{3}{20} + \frac{2}{5} \cdot \frac{11}{20} = 1.$$

Let C be a European call option with payoff $C_1 = \max(S_1 - K, 0)$ and strike price $K = 110$. The payoff at time $t = 1$ is the following:

$$C_1(i) = \begin{cases} 90 & \text{if } i = 1 \\ 10 & \text{if } i = 2 \\ 0 & \text{if } i = 3 \end{cases}$$

The price of the call option C_0 computed through $Q^{0.2}$ is:

$$C_0 = \mathbb{E}_{Q^{0.2}}(C_1) = 90 \cdot \frac{6}{20} + 10 \cdot \frac{3}{20} = \frac{57}{2} = 28.5.$$

Let be $\alpha = 0.9$. The equivalent martingale measure is $Q^{0.9} = \left(\frac{3}{80}, \frac{27}{40}, \frac{23}{80}\right)$. Also in this case $Q^{0.9} \in \mathcal{Q}$ since $\mathbb{E}_{Q^{0.9}}\left(\frac{S_1}{S_0}\right) = 1$, and the price of the call option computed trough $Q^{0.9}$ is:

$$C_0 = \mathbb{E}_{Q^{0.9}}(C_1) = 90 \cdot \frac{3}{80} + 10 \cdot \frac{27}{40} = \frac{81}{8} = 10.125.$$

♦

The trinomial model can be completed by adding another risky asset. We denote risky assets as $S^{(1)}$ and $S^{(2)}$, each of them with price process as in (11), with parameters u_i, m_i, d_i, for $i = 1, 2$. The model is complete as $K + 1 = 3 = n$, with a unique solution for q_1, q_2, q_3 (for details see Pascucci and Runggaldier, 2011).

We stress that any n-nomial market model composed by K risky asset is incomplete, for $n \geq 3$ and $K < (n - 1)$, as explained in Cinfrignini et al. (2021).

Anyhow completing the market is not always possible or desirable. Our approach would deal with a subset $\mathcal{Q}' \subseteq \mathcal{Q}$, possibly with an equality. Pricing with $\mathcal{Q}'$ would allow to model frictions in the market in the form of bid-ask spreads. The intuitive way to face the problem of frictions in a trinomial model is to define the interval of derivative's price induced by $\mathcal{Q}'$. It means that we look for the lower and the upper bounds of price, defined as:

$$\underline{X_0} = (1+r)^{-1} \inf_{Q^\alpha \in \mathcal{Q}'} \mathbb{E}_{Q^\alpha}(X_1), \quad \overline{X_0} = (1+r)^{-1} \sup_{Q^\alpha \in \mathcal{Q}'} \mathbb{E}_{Q^\alpha}(X_1). \tag{16}$$

Thus, we could look for a lower/upper pricing rule which is given by the lower/upper envelope of a class of expectations with respect to each $Q^\alpha \in \mathcal{Q}'$ and extends the fixed lower/upper price assessment.

3 Non-additive measures and non-linear functionals

When uncertainty is not quantifiable in a single probability measure and we have to deal with a set of them, we are facing a situation called *ambiguity*. Since working with the whole class of probabilities is hard, we usually consider the envelopes of the class. For instance, in the trinomial model just defined, we would consider a lower pricing rule expressed by a functional of an envelope of the set of equivalent martingale measures. In particular, in what follows, we work with the *lower* envelope, but we point out that the upper envelope leads to the same results, since they are conjugate functions. Generally, envelopes of a set of probability measures are no longer probabilities. Hence, we have to introduce generalized functions that lose the additive property: for that they are called non-additive measures. Moreover, in particular settings, there exists a link between the envelopes of linear functionals defined with respect to a class of probability measures and a non-linear functionals computed with respect to a non-additive measure, as we show in this section.

Let $(\Omega, \mathcal{F})$ be the finite space defined in the previous section, with $\mathcal{F} = \mathcal{P}(\Omega)$.

Definition 3.1 A function $\nu : \mathcal{F} \to \mathbb{R}$ is called a *non-additive measure* or a *capacity* if it is:

(i) *normalized*: $\nu(\emptyset) = 0$ and $\nu(\Omega) = 1$;

(ii) *monotone*: $\nu(A) \leq \nu(B)$ for all $A, B \in \mathcal{F}$, with $A \subseteq B$.

Moreover, a capacity ν is called:

(a) *2-monotone* or *convex* capacity if, for every $A, B \in \mathcal{F}$:

$$\nu(A \cup B) \geq \nu(A) + \nu(B) - \nu(A \cap B); \tag{17}$$

(b) *totally monotone* capacity or *belief function* (usually denoted by Bel) if, for every $A_1, \ldots, A_k \in \mathcal{F}$ with $k \geq 2$, it holds that:

$$\nu\left(\bigcup_{i=1}^{k} A_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} \nu\left(\bigcap_{i \in I} A_i\right); \tag{18}$$

(c) (coherent) *lower probability* if there exists a set $\mathcal{P}$ of probability measures on $\mathcal{F}$ such that, for every $A \in \mathcal{F}$:

$$\nu(A) = \inf_{P \in \mathcal{P}} P(A); \tag{19}$$

(d) *probability measure* if $\nu(A \cup B) = \nu(A) + \nu(B)$, for every disjoint $A, B \in \mathcal{F}$.

If ν is a belief function, then it is also a 2-monotone capacity and a (coherent) lower probability. In turn, if ν is a probability measure, then it is also a belief function. Conversely, the property of being a lower probability does not imply 2-monotonicity and, so, neither total monotonicity.

We denote by $\mathbf{V}(\Omega, \mathcal{F})$ and $\mathbf{B}(\Omega, \mathcal{F})$, respectively, the set of all capacities and that of all belief functions on $(\Omega, \mathcal{F})$, and we stress that $\mathbf{P}(\Omega, \mathcal{F}) \subseteq \mathbf{B}(\Omega, \mathcal{F}) \subseteq \mathbf{V}(\Omega, \mathcal{F})$[2].

For every 2-monotone capacity there exists a set of dominating probability measures called *core* (or credal set) (Gilboa and Schmeidler, 1994; Walley, 1991):

$$\mathbf{core}(\nu) = \{P \in \mathbf{P}(\Omega, \mathcal{F}) | P(A) \geq \nu(A),\ \forall A \in \mathcal{F}\}. \tag{20}$$

A coherent lower probability $\underline{P}$ is such that $\mathbf{core}(\underline{P}) \neq \emptyset$ and $\underline{P}$ is its lower envelope: $\underline{P}(A) = \min\limits_{P \in \mathbf{core}(\underline{P})} P(A),\ \forall A \in \mathcal{F}$. In turn, a belief function, as it is a particular lower probability, can be regarded as the lower envelope of its core:

$$Bel(A) = \min_{P \in \mathbf{core}(Bel)} P(A). \tag{21}$$

Every capacity ν can be characterized in terms of another function called *Möbius inverse* (Chateauneuf and Jaffray, 1989):

$$m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \nu(B), \qquad \nu(A) = \sum_{B \subseteq A} m(B). \tag{22}$$

Proposition 3.1 (Chateauneuf and Jaffray, 1989). Given a function $\nu : \mathcal{F} \to \mathbb{R}$, let m be its Möbius inverse. Then:

2 Every capacity ν has a conjugate function called *dual capacity*. In general it is defined as $\overline{\nu}(A) = 1 - \nu(A^C), \forall A \in \mathcal{F}$. The dual of a lower probability is said *upper probability*; the dual of a 2-monotone (convex) capacity is said 2-*alternating (concave)* capacity; the dual of a belief function is said *plausibility* function (Pl); the dual of a probability is itself.

(a) ν is a capacity if and only if:
$m(\emptyset) = 0$,
$\sum_{B \in \mathcal{F}} m(B) = 1$, and
$\sum_{\{i\} \in B \subseteq A} m(B) \geq 0$, for all $A \in \mathcal{F}$ and for all $i \in A$;

(b) ν is a 2-monotone capacity if and only if condition (a) holds and $\forall A \in \mathcal{F}$, and $\{i, j\} \in A$ with $i \neq j$, $\sum_{\{i,j\} \subseteq B \subseteq A} m(B) \geq 0$;

(c) ν is a belief function if and only if condition (a) holds and m is non-negative;

(d) ν is a probability measure if and only if condition (a) holds, m is non-negative and can be positive only on singletons.

Definition 3.2 (Gilboa and Schmeidler, 1994). Given a capacity ν and a random variable $X \in \mathbb{R}^{\Omega}$, the *Choquet expectation* of X with respect to ν, denoted by $\mathbb{C}_\nu(X)$, is defined through the Choquet integral:

$$\begin{aligned}\mathbb{C}_\nu(X) &= \oint_{\Omega} X \,\mathrm{d}\nu = \\ &= \int_0^{\infty} \nu(\{i \in \Omega | X(i) \geq x\}) \,\mathrm{d}x + \int_{-\infty}^{0} \left[\nu(\{i \in \Omega | X(i) \geq x\}) - \nu(\Omega)\right] \,\mathrm{d}x.\end{aligned} \tag{23}$$

We point out that the Choquet expectation coincides with the expected value if ν is additive (i.e. it is a probability measure P): $\mathbb{C}_\nu(X) = \mathbb{E}_P(X)$. Assuming $\Omega = \{1, \ldots, n\}$, the Choquet integral can be computed in the following way:

$$\mathbb{C}_\nu(X) = \sum_{i=1}^{n} \left[X(\sigma(i)) - X(\sigma(i+1))\right] \nu(E_i^\sigma), \tag{24}$$

where σ is a permutation of Ω such that $X(\sigma(1)) \geq \ldots \geq X(\sigma(n))$, $E_i^\sigma = \{\sigma(1), \ldots, \sigma(i)\}$, for $i = 1, \ldots, n$, and $X(\sigma(n+1)) = 0$. Moreover, for every $\nu \in \mathbf{V}(\Omega, \mathcal{F})$ with corresponding Möbius inverse m, and $X \in \mathbb{R}^{\Omega}$, the Choquet expectation of X with respect to ν can be computed through the Möbius inverse:

$$\mathbb{C}_\nu(X) = \sum_{B \in \mathcal{F} \setminus \{\emptyset\}} m(B) \min_{i \in B} X(i). \tag{25}$$

We summarize some properties of the Choquet integral:

(i) for all $A \subseteq \Omega$ we have that $\mathbb{C}_\nu(\mathbf{1}_A) = \nu(A)$, with $\mathbf{1}_A : \Omega \to \{0, 1\}$ the indicator function of A such that $\mathbf{1}_A(i) = 1$ if $i \in A$ and $\mathbf{1}_A = 0$ otherwise;

(ii) for any capacities $\nu, \varphi, \in \mathbf{V}(\Omega, \mathcal{F})$ and $\alpha, \beta \in \mathbb{R}$, it holds that
$\mathbb{C}_{\alpha\nu + \beta\varphi}(X) = \alpha \mathbb{C}_\nu(X) + \beta \mathbb{C}_\varphi(X)$;

(iii) (non-negative homogeneity) for any capacity ν and all $\alpha \geq 0$, it holds that $\mathbb{C}_\nu(\alpha X) = \alpha \mathbb{C}_\nu(X)$;

(iv) (constant additivity) for any capacity ν and all $\alpha \in \mathbb{R}$, it holds that
$\mathbb{C}_\nu(\alpha + X) = \alpha + \mathbb{C}_\nu(X)$;

(v) (monotonicity) for any capacity ν and for any $X, Y \in \mathbb{R}^{\Omega}$ such that $X \leq Y$, it holds that $\mathbb{C}_\nu(X) \leq \mathbb{C}_\nu(Y)$;

(vi) if ν is a 2-monotone capacity, for any $X, Y \in \mathbb{R}^\Omega$, the Choquet integral is super-additive: $\mathbb{C}_\nu(X+Y) \geq \mathbb{C}_\nu(X) + \mathbb{C}_\nu(Y)$, and, for every $X \in \mathbb{R}^\Omega$, the Choquet expectation equals the lower expectation with respect to the $\mathbf{core}(\nu)$ (Gilboa and Schmeidler, 1994):

$$\mathbb{C}_\nu(X) = \min_{P \in \mathbf{core}(\nu)} \sum_{i \in \Omega} P(\{i\}) X(i) = \min_{P \in \mathbf{core}(\nu)} \mathbb{E}_P(X); \tag{26}$$

(vii) if ν reduces to a belief function Bel, for any $X_1, \ldots, X_k \in \mathbb{R}^\Omega$, the Choquet integral is completely monotone:

$$\mathbb{C}_{Bel}\left(\bigvee_{i=1}^k X_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} \mathbb{C}_{Bel}\left(\bigwedge_{i \in I} X_i\right). \tag{27}$$

We stress that property (vi) continues to hold if ν reduces to a belief function and it can be interpreted as a *lower expectation*. This shows that Choquet expectation with respect to a 2-monotone capacity or a belief function leads to a specific functional inside the class of envelopes of expectations.

In the following example we show that, despite the lower envelope of a set $\mathcal{P}$ of probability measures is 2-monotone (or even a belief function), the corresponding Choquet expectation may not coincide with the lower expectation induced by $\mathcal{P}$ if its closed convex hull does not coincide with $\mathbf{core}(\underline{P})$.

Example 3.1 *Let $\Omega = \{1,2,3\}$ and X be a random variable that assumes the following values: $X(i) = i$, for $i = 1,2,3$. Let $\mathcal{P}$ be a set of three probability measures: $\mathcal{P} = \{P_1, P_2, P_3\}$ taking values reported below:*

$\mathcal{F}$	$\emptyset$	1	2	3	12	13	23	Ω
P_1	0	1/2	1/4	1/4	3/4	3/4	1/2	1
P_2	0	1/3	1/3	1/3	2/3	2/3	2/3	1
P_3	0	2/5	2/5	1/5	4/5	3/5	3/5	1

The lower probability $\underline{P}(A) = \min_{P \in \{P_1,P_2,P_3\}} P(A), \forall A \in \mathcal{F}$, and its Möbius inverse m are reported in the following table:

$\mathcal{F}$	$\emptyset$	1	2	3	12	13	23	Ω
P_1	0	1/2	1/4	1/4	3/4	3/4	1/2	1
P_2	0	1/3	1/3	1/3	2/3	2/3	2/3	1
P_3	0	2/5	2/5	1/5	4/5	3/5	3/5	1
$\underline{P}$	0	1/3	1/4	1/5	2/3	3/5	1/2	1
m	0	1/3	1/4	1/5	1/12	1/15	1/20	1/60

Since $m(A) \geq 0$ for every $A \in \mathcal{F}$, the lower probability $\underline{P}$ is a belief function.

The Choquet integral of X with respect to $\underline{P}$, generally, is not equal to the lower expectation of X computed among $P \in \mathcal{P}$ since the convex hull $\mathbf{conv}(\mathcal{P})$, which is closed as $\mathcal{P}$ is finite, does not coincide with $\mathbf{core}(\underline{P})$. To show that, we compute the extreme points of $\mathbf{core}(\underline{P})$. Extreme points of $\mathbf{core}(\underline{P})$ are computed in the following way: for any permutation of indices $\sigma = (\sigma(1), \ldots, \sigma(n))$, extreme points are computed as $P^\sigma = (P(\sigma(1)), \ldots, P(\sigma(n)))$ with $P(\sigma(i)) = \underline{P}(\{\sigma(1), \ldots, \sigma(i)\}) - \underline{P}(\{\sigma(1), \ldots, \sigma(i-1)\})$.

Hence, for any permutation of $\{1,2,3\}$, we have the following set of extreme points $\mathrm{ext}(\mathbf{core}(\underline{P}))$:

$$P^{(1,2,3)} = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right) = P_2, \qquad P^{(1,3,2)} = \left(\tfrac{1}{3}, \tfrac{2}{5}, \tfrac{4}{15}\right),$$

$$P^{(2,1,3)} = \left(\tfrac{5}{12}, \tfrac{1}{4}, \tfrac{1}{3}\right), \qquad P^{(2,3,1)} = \left(\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}\right) = P_1,$$

$$P^{(3,1,2)} = \left(\tfrac{2}{5}, \tfrac{2}{5}, \tfrac{1}{5}\right) = P_3, \qquad P^{(3,2,1)} = \left(\tfrac{1}{2}, \tfrac{3}{10}, \tfrac{1}{5}\right).$$

This proves that $\mathbf{conv}(\mathcal{P}) \neq \mathbf{core}(\underline{P})$*, hence the lower expectation with respect to* $\mathcal{P}$ *and the Choquet integral with respect to* $\underline{P}$*, generally, lead to different results:*

$$\underline{\mathbb{E}}(X) = \min_{P \in \{P_1, P_2, P_3\}} \mathbb{E}_P(X) = \min\{1.75, 2, 1.8\} = 1.75,$$

$$\mathbb{C}_{\underline{P}}(X) = (3-2)\underline{P}(3) + (2-1)\,\underline{P}(23) + (1-0)\underline{P}(\Omega) = 1.7.$$

Therefore, we get that $\mathbb{C}_{\underline{P}}(X) < \underline{\mathbb{E}}(X)$*, since* $\mathbf{conv}(\mathcal{P}) \subset \mathbf{core}(\underline{P})$. ♦

At this point the question that arises is if a n-nomial model leads to analogous results. This problem has been faced in Cinfrignini et al. (2021). It is proved that any n-nomial market model, for $n \geq 3$ is incomplete and the lower envelope of the set of equivalent martingale measures is a belief function but the closure of the set of equivalent martingale measures does not coincide with the core of its lower envelope.

4 Non-linear pricing rules

Let us consider a one-period financial market with frictions in the form of bid-ask spreads, that can be due to the presence of intermediaries, taxes, or to the incompleteness of the market. The market consists of a risk-free bond B and of a set of K risky assets with payoffs $S_1^{(1)}, \ldots, S_1^{(K)}$.

For $k = 1, \ldots, K$, each asset's price is defined through an interval $[\underline{S}_0^{(k)}, \overline{S}_0^{(k)}]$, where $\underline{S}_0^{(k)}$ is called *bid price* and $\overline{S}_0^{(k)}$ is called *ask price* (it is tacit that $\underline{S}_0^{(k)} \leq \overline{S}_0^{(k)}$ where equality holds only if the k-th asset is frictionless). The bond price process is $(B_0 = 1, B_1 = 1 + r)$ and it is frictionless, i.e. $\underline{B}_0 = \overline{B}_0 = B_0$.

The problem is to determine non-linear functionals able to characterize bid and ask prices. In Acciaio et al. (2016), for instance, lower and upper expectations are used as non-linear functionals. The question that in literature has been addressed is if non-linear functionals can be defined by means of the lower/upper expectation with respect to a set of probabilities, or by means of a Choquet integral with respect to a 2-monotone capacity or a belief function, and if the two approaches give out to the same outcome.

In what follows we will see that the same question arises in the trinomial model where a generalization of no-Dutch book condition can be shown to hold. We point out that the framework in Chateauneuf et al. (1996) and Coletti et al. (2020) is from the upper price point of view, in terms of concave capacities and plausibility functions. Here, the setting has been reversed in terms of convex capacities and belief functions. Original results do not change since concave (plausibility) functions are the conjugate of convex (belief) functions.

We consider a *global* lower pricing rule $\underline{\pi} : \mathbb{R}^\Omega \to \mathbb{R}$ defined for all $X_1 \in \mathbb{R}^\Omega$ as:

$$\underline{\pi}\,(X_1) = \underline{X}_0, \tag{28}$$

which is not assumed to be linear.

In accordance with Chateauneuf et al. (1996)[3], we make the following assumptions:

(A1) *monotonicity*: for $X_1, Y_1 \in \mathbb{R}^\Omega$, if $X_1 \geq Y_1$ then $\underline{\pi}\,(X_1) \geq \underline{\pi}\,(Y_1)$;

(A2) *frictionless bond*: there is a risk-free bond $B_0 = 1, B_1 = 1+r$ that is not frictional $\underline{\pi}(\alpha B_1) = \alpha$, for all $\alpha \in \mathbb{R}$;

(A3) *super-additivity*: for $X_1, Y_1 \in \mathbb{R}^\Omega$, we have that $\underline{\pi}\,(X_1) + \underline{\pi}\,(Y_1) \leq \underline{\pi}\,(X_1 + Y_1)$, with $i, j \in \{1, \ldots, K\}$, where equality holds only if X_1 and Y_1 are *comonotone*[4].

3 In the quoted paper, the authors consider already discounted amounts that, in our setting, is equivalent to take $r = 0$.

4 Two assets $X_1, Y_1 \in \mathbb{R}^\Omega$ are comonotone if they vary in the same way: $\forall \omega, \omega' \in \Omega$, $[X_1(\omega) - X_1(\omega')][Y_1(\omega) - Y_1(\omega')] \geq 0$.

Theorem 4.1 (Chateauneuf et al., 1996). Under assumptions (A1)–(A3), for all $X_1 \in \mathbb{R}^\Omega$ there exists a unique convex capacity ν such that the *global* lower pricing rule $\underline{\pi}$ can be expressed as a discounted Choquet expectation of the payoff with respect to ν:

$$\underline{\pi}(X_1) = (1+r)^{-1} \oint_\Omega X_1 \, \mathrm{d}\nu = (1+r)^{-1} \mathbb{C}_\nu (X_1) . \tag{29}$$

We stress that the Choquet integral with respect to a 2-monotone capacity is equivalent to the lower expectation with respect to the set of probability measures in $\mathbf{core}(\nu)$, hence the bid price is equivalently computed as:

$$\underline{X}_0 = \underline{\pi}(X_1) = (1+r)^{-1} \min_{P \in \mathbf{core}(\nu)} \mathbb{E}_P (X_1) . \tag{30}$$

As we already pointed out, the same model can be set up from the *upper* point of view with respect to a concave capacity, replacing assumption (A3) with *sub-additivity* property (that is the version in Chateauneuf et al., 1996). Since it is the dual function of a convex capacity, for each $X_1 \in \mathbb{R}^\Omega$, we can compute the ask price $\overline{X}_0$ as a discounted Choquet integral with respect to the conjugate concave capacity, which can be expressed in terms of an upper pricing rule:

$$\overline{X}_0 = \overline{\pi}(X_1) = -\underline{\pi}(-X_1). \tag{31}$$

The approach of Chateauneuf et al. (1996) characterizes a lower pricing rule already defined on the whole $\mathbb{R}^\Omega$. If we refer to the K fixed risky assets and identify the payoff of the risk-free bond B_1 with a 0-th asset, and $-B_1$ with a $(K+1)$-th asset, we have a finite set of payoffs $\mathcal{G} = \{S_1^{(0)}, \ldots, S_1^{(K+1)}\}$. In this case, we have a lower price assessment $\underline{\pi} : \mathcal{G} \to \mathbb{R}$ such that $\underline{\pi}\left(S_1^{(k)}\right) = \underline{S}_0^{(k)}$, for $k = 1, \ldots, K$, $\underline{\pi}\left(S_1^{(0)}\right) = 1$ and $\underline{\pi}\left(S_1^{(K+1)}\right) = -1$. Now, our goal is to find a lower pricing rule $\underline{\pi}' : \mathbb{R}^\Omega \to \mathbb{R}$ that extends $\underline{\pi}$ and can be expressed as a discounted Choquet expectation. This problem can be tackled in the framework of belief functions by relying on results given in Coletti et al. (2020).

If there exists a belief function $Bel : \mathcal{F} \to [0,1]$ such that, for $k = 0, \ldots, K+1$, the lower price assessment is defined as the discounted Choquet expectation with respect to Bel, that is it satisfies:

$$\underline{\pi}\left(S_1^{(k)}\right) = \mathbb{C}_{Bel}\left(\tilde{S}_1^{(k)}\right), \tag{32}$$

then the lower price assessment is called *CBel-coherent*. As usual, $\tilde{S}_1^{(k)}$ denotes the discounted payoff, for $k = 0, \ldots, K+1$.

Theorem 4.2 (Coletti et al., 2020). For a finite $\mathcal{G}$ defined as above the following statements are equivalent:

(i) $\underline{\pi}$ is a CBel-coherent price assessment;

(ii) $\underline{\pi}$ avoids *CBel-Dutch book* opportunities: for every $\boldsymbol{\lambda} \in \mathbb{R}^{K+2}$, the following condition holds:

$$\max_{B \in \mathcal{F} \setminus \{\emptyset\}} \sum_{k=0}^{K+1} \lambda_k \left(\min_{i \in B} \tilde{S}_1^{(k)}(i) - \underline{\pi}\left(S_1^{(k)}\right)\right) \geq 0. \tag{33}$$

Condition (33) assures that there cannot be a portfolio that leads to a sure loss, defined under *partially resolving uncertainty* (Jaffray, 1989), i.e., working over $\mathcal{F} \setminus \{\emptyset\}$.

The no-Dutch book condition in the setting of belief functions in Equation (33) is weaker than the classical no-Dutch book condition in Equation (3) since in the latter case we are working under *completely resolving uncertainty*. Completely resolving uncertainty is the common assumption of the classical Dutch-book condition (Equation (3)) and requires that, once uncertainty is resolved, the knowledge of the true state $i \in \Omega$ will be acquired. Conversely, under *partially resolving uncertainty* we assume that, when uncertainty is resolved, we may acquire the information that an event B has occurred but we may not identify the state $i \in B$ that turns out to be true. In particular, condition (33) considers a systematically pessimistic behavior as, for every $k = 0, \ldots, K+1$, we take the minimum payoff given by all $i \in B$, defined as $\min\limits_{i \in B} \tilde{S}_1^{(k)}(i)$. We notice that Theorem 4.2 does not guarantee the uniqueness of Bel and, so, of the lower pricing ruler $\underline{\pi}'$ extending $\underline{\pi}$. Nevertheless, every such extension satisfies conditions (A1)–(A3) introduced before.

We finally get back to the trinomial market model. In Cinfrignini et al. (2021) it is proved that the lower probability $\underline{Q}$ of the set $\mathcal{Q}$ of equivalent martingale measures, computed as $\underline{Q}(A) = \min\limits_{Q \in \mathrm{cl}(\mathcal{Q})} Q(A)$, $\forall A \in \mathcal{F}$, is a belief function. Therefore, this would suggest to define a lower pricing rule as the discounted Choquet expectation with respect to $\underline{Q}$. Unfortunately, a situation analogous to Example 3.1 occurs since the closure $\mathrm{cl}(\mathcal{Q})$ does not coincide with $\mathbf{core}(\underline{Q})$. Thus the discounted Choquet expectation does not coincide with the lower expectation computed with respect to $\mathcal{Q}$, and so the two approaches lead to different results.

We also notice that, still referring to the trinomial model, by considering the set $\mathcal{G} = \{B_1, S_1, -B_1\}$, with the lower pricing assessment defined as $\underline{\pi}(B_1) = 1$, $\underline{\pi}(S_1) = S_0$, and $\underline{\pi}(-B_1) = -1$, we get that the no-Dutch book condition in *(ii)* of Theorem 4.2 holds. It is actually possible to show that $\underline{\pi}$ can be extended by a discounted Choquet expectation functional computed with respect to a non-additive belief function Bel which, however, must be different from $\underline{Q}$.

5 Conclusion

In this paper we have presented a survey on classical pricing theory and we focused on markets with frictions in the form of bid-ask spreads. Frictions are largely proved to exists and are studied in order to embody them into price models. Then, after having introduced non-additive measures and the Choquet expectation, we recalled the properties characterizing a global lower pricing rule defined as the discounted Choquet expectation with respect to a convex capacity (Chateauneuf et al., 1996). Then, referring to a finite set of payoffs, we showed a condition that guarantees the representation of lower prices as discounted Choquet expectation with respect to a belief function. The latter condition is in the form of no-Dutch book under partially resolving uncertainty. We also showed that the lower envelope of equivalent martingale measures in the trinomial model does not produce sharp lower prices, with respect to the class of martingale measures, if used to compute discounted Choquet expectations (see Cinfrignini et al., 2021). Nevertheless, the lower prices of fixed securities satisfy the generalized no-Dutch book condition given in Coletti et al. (2020).

References

Acciaio B., Beiglböck M., Penkner F. and Schachermayer W. (2016), A model-free version of the fundamental theorem of asset pricing and the super-replication theorem, *Mathematical Finance*, 26 (2), 233-251.

Allingham M. (1991), *Arbitrage. Elements of Financial Economics*, Palgrave Macmillan, London.

Amihud Y. and Mendelson H. (1986), Asset pricing and the bid-ask spread, *Journal of Financial Economics*, 17 (2), 223-249.

Amihud Y. and Mendelson H. (1991), Liquidity, Maturity, and the Yields on U.S. Treasury Securities, *The Journal of Finance*, 46 (4), 1411-1425.

Bensaid B., Lesne J-P., Pagès H. and Scheinkman J. (1992), Derivative asset pricing with transaction costs, *Mathematical Finance*, 2 (2), 63-86.

Bertsimas D., Kogan L. and Lo A. W. (2001), Hedging Derivative Securities and Incomplete Markets: An ϵ-Arbitrage Approach, *Operations Research*, 49 (3), 372–397.

Capotorti A., Coletti G. and Vantaggi, B. (2008), Preferences Representable by a Lower Expectation: Some Characterizations, *Theory and Decision*, 64, 119–146.

Cerný A. (2009), *Mathematical Techniques in Finance: Tools for Incomplete Markets*. Princeton: Princeton University Press.

Cerreia-Vioglio S., Maccheroni F. and Marinacci M. (2015), Put–Call Parity and market frictions, *Journal of Economic Theory*, 157 (C), 730-762.

Chateauneuf A. and Cornet B. (2022), Submodular financial markets with frictions, *Economic Theory*, 73, 721–744.

Chateauneuf A. and Jaffray J.-Y. (1989), Some characterizations of lower probabilities and other monotone capacities through the use of Möbius inversion, *Mathematical Social Sciences*, 17 (3), 263-283.

Chateauneuf A., Kast R. and Lapied A. (1996), Choquet Pricing For Financial Markets With Frictions, *Mathematical Finance*, 6 (3), 323-330.

Chen Z. and Kulperger R. (2006), Minimax pricing and Choquet pricing, *Insurance: Mathematics and Economics*, 38 (3), 518-528.

Cinfrignini A., Petturiti D. and Vantaggi B. (2021), Envelopes of equivalent martingale measures and a generalized no-arbitrage principle in a finite setting, *arXiv:2107.01240*

Coletti G., Petturiti D. and Vantaggi B. (2016), Conditional belief functions as lower envelopes of conditional probabilities in a finite setting, *Information Sciences*, 339, 64-84.

Coletti G., Petturiti D. and Vantaggi B. (2020), A Dutch book coherence condition for conditional completely alternating Choquet expectations, *Bollettino dell'Unione Matematica Italiana*, 13, 585–593.

Cox J.C., Ross S.A. and Rubinstein M. (1979), Option pricing: A simplified approach, *Journal of Financial Economics*, 7, 229-263.

Cozman F.G. (2000), Credal networks, *Artificial Intelligence*, 120 (2), 199-233.

Dempster A.P. (1967), Upper and Lower Probabilities Induced by a Multivalued Mapping, *The Annals of Mathematical Statistics*, 38 (2), 325-339.

Dybvig P.H. and Ross S. (1989), *Arbitrage*, in Eatwell J., Milgate M. and Newman P., *Finance*. Palgrave Macmillan, London.

Erreygers A., De Bock J., de Cooman G. and Van Camp A. (2019), Optimal control of a linear system subject to partially specified input noise, *International Journal of Robust and Nonlinear Control*, 29, 3892–3914.

Etner J., Jeleva M. and Tallon J-M. (2012), Decision Theory Under Ambiguity, *Journal of Economic Surveys*, 26 (2), 234-270.

Ghirardato P. and Marinacci M. (2001), Risk, Ambiguity, and the Separation of Utility and Beliefs, *Mathematics of Operations Research*, 26 (4), 864–890.

Gilboa I. and Marinacci M. (2011), Ambiguity and the Bayesian Paradigm, *Working Papers 379*, IGIER, Bocconi University.

Gilboa I. and Schmeidler D. (1989), Maxmin expected utility with non-unique prior, *Journal of Mathematical Economics*, 18 (2), 141-153.

Gilboa I. and Schmeidler D. (1994), Additive Representations of Non-Additive Measures and the Choquet Integral, *Annals of Operations Research*, 52 (1), 43-65.

Jaffray J-Y. (1989), Coherent Bets Under Partially Resolving Uncertainty and Belief Functions, *Theory and Decision*, 26 (2), 99-105.

Jouini E. and Kallal H. (1995), Martingales and Arbitrage in Securities Markets with Transaction Costs, *Journal of Economic Theory*, 66, 178-197.

Pascucci A. and Runggaldier W.J. (2011), *Financial Mathematics. Theory and problems for multi-period models*, Springer Verlag, Italia, Series UNITEXT.

Petturiti D. and Vantaggi B. (2020), Modeling agent's conditional preferences under objective ambiguity in Dempster-Shafer theory, *International Journal of Approximate Reasoning*, 119, 151-176.

Petturiti D. and Vantaggi B. (2022), Conditional decisions under objective and subjective ambiguity in Dempster-Shafer theory, *Fuzzy Sets and Systems*, 447, 155-181.

Pliska S. (1997), *Introduction to Mathematical Finance: Discrete Time Models*, (2nd ed.), Malden: Blackwell, Oxford.

Runggaldier W. (2006), Portfolio optimization in discrete time, *Accademia delle Scienze dell'Istituto di Bologna*.

Schervish M.J., Seidenfeld T. and Kadane J.B. (2008), The Fundamental Theorems of Prevision and Asset Pricing, *International Journal of Approximate Reasoning*, 49 (1), 148-158.

Schmeidler D. (1986), Integral representation without additivity, *Proceedings of the American Mathematical Society*, 97 (2), 255–261.

Schmeidler D. (1989), Subjective Probability and Expected Utility without Additivity, *Econometrica*, 57 (3), 571–587.

Shafer G. (1976), *A Mathematical Theory of Evidence*. Princeton: Princeton University Press.

T'Joens N., De Bock J. and de Cooman G. (2021), Game-theoretic upper expectations for discrete-time finite-state uncertain processes, *Journal of Mathematical Analysis and Applications*, 504 (2).

Walley P. (1991), *Statistical Reasoning with Imprecise Probabilities*. Chapman & Hall, London.