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Research article

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THE ROLE OF ASYMMETRY ABOUT INVESTOR PREFERENCES

Abstract

This article is about the role of asymmetry in the distribution of portfolio returns and investors' preferences. It is well known that the skewness of the distribution can play some roles in preferences, but in economic theory this role is usually conflated with the concept of the utility function and, in particular, with expected utility maximization. This perspective seems to us unsatisfactory in two respects. First, the financial intuition about the possibility of accounting for asymmetries is too abstract and difficult for practitioners to grasp. Second, this strategy works only under some implicit conditions such as the existence of third moment; not such a weak assumption for financial returns. Here we propose a different strategy. It considers the comparison between the mean and the median of the distribution of returns. Thus, we obtain a representation that gives us an idea of the possibility of favouring a positive asymmetry and disfavoring a negative one. The main advantage of this representation is that it contains only probabilistic concepts (no utility theory) and is easily understood and communicated by practitioners. Moreover, in this way the existence of a third moment is not necessary, the first one is sufficient.

Keywords: skewness, preferences, moments, distribution.

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1 Introduction

In finance, asset allocation is one of the main problems, and investors' preferences play a crucial role about it. Indeed, the selected portfolio depends on investors' preferences. More precisely, in asset allocation, the distribution of the returns of all potential assets is important, because the distribution of the returns of the selected portfolio depends on the distribution of the assets it contains. Moreover, preferences, at least in general, imply something about the desired distribution of portfolio returns. Therefore, preferences must be linked in some way to the distributional properties of returns.

The importance of measures such as mean and variance of portfolio returns is well known, but asymmetry also plays some roles in the finance literature. However, the intuition about the financial role of mean and variance is trivial, while that about asymmetry is not.

In particular, the desire for the highest possible mean of returns combined with the lowest possible variance is usually considered a necessary condition for an investor to be considered *rational* and *risk averse*. Moreover, this is the starting point for justifying the mean-variance model. This model can be considered as the natural statistical counterpart of the financial concept of return-risk optimization.

Unfortunately, the role of asymmetry is less clear.

Following the paradigm of utility theory, more precisely the criteria of expected utility maximization, the preceding considerations on preferences with respect to mean and variance can be translated into analytical conditions for the *utility function* of the investor [$U(r_p)$]. More precisely, it was shown that the necessary conditions are: positive the first derivative and negative the second derivative. It should be emphasised, however, that this paradigm and its associated conditions are not mandatory to justify a mean-variance model; statistical/financial intuition is sufficient.

In any case, the paradigm of expected utility maximization is the most widespread justification strategy and brings readers to this kind of representation[1]:

$$\mathrm{E}[U(\mathrm{r_p})] = \int_{-\infty}^{+\infty} U(\mathrm{r_p})dF(\mathrm{r_p}) = \sum_{n=0}^{\infty} \frac{U^{(n)}(E[\mathrm{r_p}])}{n!} E(\mathrm{r_p} - E[\mathrm{r_p}])^n$$

On the right side, the Taylor expansion series rule around the expected value of r_p (portfolio return) is used. The randomness of r_p leads us to use its distribution [$F(r_p)$], while the Taylor expansion series rule leads us to reduce all relevant features to its moments.

It has been argued that negative asymmetry is an undesirable feature for the distribution of portfolio returns, while positive asymmetry is desirable. In the context of the Taylor expansion series, it was shown that this type of preference is consistent with the positive third derivative of the utility function[2].

Unfortunately, it is not easy to grasp the practical content of this analytical condition. Indeed, sometimes the positive asymmetry is preferred without really understanding why this is a reasonable choice. The most common misunderstandings among

[1] For explanations that lead to representations like above, see: Jurczenko and Maillet (2006) section 1.2; Fabozzi et al. (2006) pp.131-137; Jondeau et al. (2007) p. 350.

[2] This result can be generalized. Indeed, it was shown that for risk averse investors skewness should be maximized and kurtosis minimized. In general, odd moments should be maximized and the even ones minimized. Most influential contribution about that, is in: Scott and Horvath (1980).

practitioners arise from the confusion of concepts like: negative asymmetry, lower mean, higher variability and/or risk. For example, there is a misconception that negative asymmetry implies something like a reduction in expected value and/or an increase in the variability of returns. Such reasoning is contradictory because the de-sirability/un-desirability of positive/negative asymmetry should be evaluated *ceteris paribus*, primarily keeping the mean and variance of portfolio returns constant. More problematic discussions are related to the concept of risk and the impact of asymmetry on it. Most of the problems arise from trying to evaluate this effect without even defining the metric of risk.

On the academic side, there is a tendency to consider the undesirability of the negative asymmetry as valid, without dwelling on justifications, and/or to assume the recognition of the positive third derivative of the utility function in question. However, this practice does not seem to be satisfactory on a logical level, because the utility function should encode (predefined) preferences and not suggest/imply them.

In addition, there are other, more technical points. The preceding exposition is not always admissible, since it requires relevant assumptions that are usually passed over in the financial literature. Indeed, the Taylor expansion series rule is reliable only under certain conditions. This rule is also often used in the form of an approximation, but this possibility and the quality of the approximation depend on both the properties of the utility function $U(r_p)$ and the distribution function $F(r_p)$. We cannot analyze them exhaustively here, but some points should be highlighted[3]. First of all, the condition of the positive third derivative mentioned earlier implies that it exists and thus $U(r_p)$ is derivable at least three times. This is not an obvious condition for utility functions and may impose undesirable/unanticipated restrictions on investor preferences. Moreover, $F(r_p)$ must admit moments, at least the same number of moments included in the approximation. Staying with the point discussed here, at least the third moment must exist. This is not such a weak assumption for financial returns, we will come back to this point later.

The following discussion attempts to show in a simple and intuitive way another possible rationale for preferences regarding asymmetry. The idea is to consider only probabilistic concepts, without referring to Utility Theory. In this way, most of the previous problems will be avoided. Moreover, we will only use concepts that can be easily transferable to financial intuition.

2 The concept of asymmetry

In general, the role of asymmetry is to give us clues about the shape of the distribution (indeed, we can refer to asymmetry and kurtosis as indices of shape). Accor-ding to the rule we will use later, we can say that asymmetry tells us something about the shape of the distribution around its median; indeed, under asymmetry, the right and left sides of the distribution are different. However, it must be emphasised that even with asymmetric distributions, the mean remains the most important measure of trend; sometimes this point is not clear enough. A lack of understanding of this fact can lead to misconceptions about the meaning of asymmetry.

It seems like a good idea to talk about a problem right away. In finance, it is common to use terms like asymmetry/skewness and/or third moment and/or index of asymmetry/skewness (by Fisher) as synonyms. This may well be true for the second and third terms, since they are only standardization problems, but not for the first one.

[3] For a detailed analysis see Loistl (1976).

Asymmetry in the distribution is a tricky concept and should not be immediately associated with the third moment (or the resulting Fisher's index), even though this is common practice in the finance literature.

2.1 Asymmetry Index of Fisher

To account for asymmetry, several skewness indices have been proposed in the literature; the most commonly used in finance is the Fisher's one. It is based on the central third moment:

$$s_p^3 = E[r_p - \mu_p]^3$$

where r_p is the portfolio return (the random variable (r.v.) under analysis), whereas μ_p stands for its mean. Skewness index by Fisher is:

$$\frac{s_p^3}{\sigma_p^3}$$

where σ_p is the standard deviation of r_p.

It is so widely and frequently used that, at least in finance, the concept of asymmetry is immediately associated with it without warning; indeed, it is commonly referred to as the "index of asymmetry/skewness". However, it is worth noting that the concept of asymmetry can lead to different measures[4]. In any case, different skewness indices have the property of becoming equal to zero when the analysed distribution is symmetric.

Unfortunately, however, even if a given index is zero, this is no guarantee of the symmetry of the analyzed distribution; this is true even for the Fisher index. It is a necessary but not sufficient condition for proving symmetry. One suggestion would be to compute multiple indices, but this is tedious and, worse, generally insufficient to prove symmetry. In any case, Fisher's skewness index is not an infallible tool for distinguishing between symmetric and asymmetric distributions. Worse, this problem is neither the only nor the most important one for this index.

The main limitation of the Fisher index is that it assumes at least the finiteness of the third moment. Unfortunately, it is worth noting that the existence of third moments for financial returns is not a weak assumption[5]. This issue is related to the problem of fat tails, a quite common and well-known property of financial returns. Therefore, Fisher's skewness index is not a good choice specifically for financial returns. In fact, we can observe that in quantitative finance some distributions are considered useful that have undefined the third moment; for example, α-stable, which never admits the third moment, or *t*-Student, which admits it only when the tail index is greater than 3.

[4] For a short review of asymmetry indexes see Piccolo (2010) pp. 157-164.

[5] For a discussion about some problems of the Fisher index of skewness see Kim and White (2004).

2.2 Comparison between mean and median

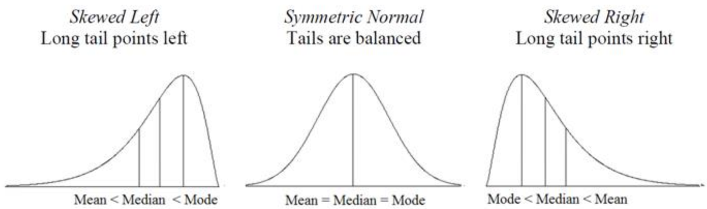
For the discussion of asymmetry, there is a method that focuses on the compari-son among mean, median, and mode. It does not require moments higher than the first one, is therefore more general than the Fisher's skewness index, and it solves at least one of the weaknesses presented earlier.

Some problems are related to multimodal distributions but financial returns usually exhibit best fitting with unimodal alternatives. Moreover, the role of mode is irrelevant in following reasoning. More in particular:

- if mean < median, distribution is negatively (left) skewed
- if mean > median, distribution is positively (right) skewed
- if mean = median, distribution could be symmetric.

It is appropriate to refer on this graph[6]:

Figure 1. Distributions: (negatively) skewed left, symmetric, (positively) skewed right



Source: Doane and Seward (2011)

Unfortunately, it has been shown that even this rule is not infallible[7]; however, it fails only in pathological cases. If we restrict our analysis to continuous and unimodal distributions, most problematic cases cannot arise; fortunately, most relevant cases in finance fit this restriction. This fact leads us to conclude that, at least for financial returns, skewness indices based on the comparison between mean and median are more reliable than those based on the third moment[8].

Shortly, we will propose an argument based on the previous rule. However, to avoid any ambiguity, it is necessary to give a definition of *symmetry in distribution*.

We can say that the distribution of a r.v. X is symmetric if:

$$p(m - x) = p(m + x) \quad \text{for any possible value } x \text{ (of } X\text{)},$$

[6] Adapted from Doane and Seward (2011) p. 3.

[7] Indeed in Piccolo (2010, p. 158) it is affirmed that "… *mentre media ≠ mediana implica certamente un'asimmetria nella distribuzione, media = mediana non implica necessariamente una simmetria*", and a related example is given (pp. 163 – 164). In other words, we can say that if the distribution is symmetric then mean = median; however, even if this equality holds, it is not a sure warranty about the symmetry of the distribution. In general, it is not possible to define the concept of symmetry from comparison between mean and median, even if sometimes, maybe for simplicity too, such a definition has been suggested. Indeed, in literature some doubts have been raised about the rule of comparison between mean and median; see von Hippel (2005).

[8] Two indexes based on these quantities are: the Hotelling and Solomon's one and the Yule and Bowley's one. See Piccolo (2010) p. 158.

where $p()$ represents the *density function* of X if X is a continuous r.v., or the *probability function* if X is discrete; m is the median of X.

Moreover, if X is continuous we can add that symmetry implies:

$$F(m - x) = 1 - F(m + x),$$

where $F()$ is the cumulative distribution function (CDF) of X.

An index of asymmetry based directly on this definition is the Bonferroni's one. It is therefore more general than the indices based on moments, however it is rarely used[9].

Moreover, it should be noted that, with the addition of some distributional hy-potheses, the limits of the various skewness indices should be discussed again. For some specified distributions used in quantitative finance, the presence of asymmetry is directly, and indisputably, revealed by some *ad hoc* parameters. An example is the *t-skew distribution*, where the sign of a specific parameter tells us the side of asymmetry, if any; if this parameter is equal to 0, t-skew distribution boils down in the *t-Student* one.

2.3 Representation based on comparison between mean and median

Having clarified the meaning of asymmetry in distribution, we can come back to its relation to investor preferences.

The financial intuition that leads investors to prefer positive (right) skewness and dislike negative (left) skewness is based on Figure 1. Indeed, we can observe that for left-skewed distributions, extremely negative realisations are more likely than the extremely positive ones; while for right-skewed distributions, extremely negative rea-lisations are less likely than the extremely positive ones.

Without loss of generality, let us now consider financial returns as distributions with zero mean: for the right skewed distributions, extreme gains are more likely than extreme losses, while for the left skewed distributions, extreme gains are less likely than extreme losses. For symmetric distributions, gains and losses, extreme or not, are equally likely. This is the financial intuition about the asymmetry of the distribution of returns. However, it is not easy to find a technical explanation for this in the finance literature; we propose here the following one.

From the comparison between mean and median, for negatively skewed distributions, the following must holds:

$$\text{“mean < median”, therefore } P(X < \mu) < P(X > \mu).$$

Now, it is useful the following decomposition:

$$E[X] = \mu = E[X|X < \mu]P(X < \mu) + E[X|X > \mu]P(X > \mu).$$

Therefore: $\mu - E[X|X < \mu]P(X < \mu) = E[X|X > \mu]P(X > \mu)$;

[9] For more information, it is possible to see Piccolo (2010) pp. 159 – 161. A test based on the Bonferroni index is discussed in Mira (1999).

then, considering for simplicity $\mu = 0$, we have that:[10]

$$|E[X|X < 0]P(X < 0)| = |E[X|X > 0]P(X > 0)|.$$

This relation is true in general, regardless equality between $P(X < 0)$ and $P(X > 0)$ (symmetry) or inequality (asymmetry). However, in case of negative asymmetry we have that:

$$|E[X|X < 0]| > |E[X|X > 0]|.$$

This representation helps us to understand that in the case of negative asymmetry, if we look at the absolute values and separate the losses from the gains, the expected losses are greater than the expected gains; in the case of positive asymmetry, the expected losses are smaller than the expected gains; under symmetry, the expected losses and gains are equal.

This proof is about expected gains/losses considered separately, not about extreme events; however, the two concepts are closely related. Indeed, if "expected losses" are larger than "expected gains", it seems sufficient to add some distributive assumptions to ensure that the same result holds for extreme events[11]. In any case, regardless of such assumptions, the result shown above can be taken as an argument that positive asymmetry is desirable and negative asymmetry is undesirable. Such an argument is easily understood by practitioners.

Now, regardless the simplified hypothesis of zero mean, we can find this form:

$$|E[X|X < \mu] - \mu| > |E[X|X > \mu] - \mu|$$

or in more concise terms:

$$\mu - \mu_- > \mu_+ - \mu$$

where $\mu_- = E[X|X < \mu]$ and $\mu_+ = E[X|X > \mu]$. In this case, more care is needed in defining "losses" and "gains"; they should be measured as the difference from the expected return (μ). However, the interpretation remains the previous one. It is also possible to check the following useful equality:

$$\frac{\mu - \mu_-}{\mu_+ - \mu} = \frac{p_+}{p_-}$$

where $p_+ = P(X > \mu)$ and $p_- = P(X < \mu)$.

This means that the relative magnitude of expected losses/gains (asymmetry ratio) depends only on the ratio between the above probabilities. The two ratios are always the same, but they are equal to 1 only under symmetry (>1 in case of negative asymmetry, <1 in case of positive asymmetry).

[10] Note that: $|-E[X|X < \mu]P(X < \mu)| = |E[X|X < \mu]P(X < \mu)|$.

[11] We do not face this problem here in a formal way, however we can give some insight. If the distributions are like in Figure 1 (well shaped distributions, then: continuous, unimodal, with monotonic decreasing tails, without truncations, etc.), result showed above for expected values (gain/loss) seems generalizable to extreme events too. Graphs suggest this conjecture.

This ratio can be considered an index of asymmetry. However, in order to achieve a more conventional reading the following simple transformation is needed:

$$s = 1 - \frac{p_+}{p_-};$$

if the distribution under analysis is left skewed $s < 0$, if it is right skewed $s > 0$, if it is symmetric $s = 0$.

Moreover, it is worth noting that under positive asymmetry it holds:
$p_+ < p_-$ and $\mu - \mu_- < \mu_+ - \mu$ (probability of loss greater but expected loss lower).

This situation, that should be the desired one, at first glance can seem strange for practitioners, due to $p_+ < p_-$ (probability of gains smaller than that of losses). However, it must hold if the three kinds of distribution (negatively skewed, positively skewed, symmetric) must share the same mean, therefore the same redditivity[12]. Indeed, the desirability of positively skewed distributions is based only on the reduction of the relative dimension of expected/extreme losses, in comparison with expec-ted/extreme gains, not on the general redditivity.

3 Conclusions

The fact that for securities/portfolios a positive asymmetry of the return distribution is better than a negative one is generally accepted in finance.

However, the reasons for this fact are sometimes misunderstood or simply igno-red by practitioners. On the academic side, the most common explanation is based on quite abstract concepts such as analytical conditions on utility functions. These ratio-nales are not easily communicated to practitioners, and their financial intuition may remain obscure. Worse, this rationale requires some implicit conditions such as the existence of third moments; not such a weak assumption for financial return distribu-tions.

The explanation presented here does not involve utility theory and related analytical conditions, but only probabilistic ones. The main results are directly transferable to financial intuition and therefore easily communicated to practitioners. Moreover, this reasoning works independently on the existence of moments greater than the first. It is therefore more general than those involving the third moments and allows one to avoid associated difficulties.

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[12] It may be noted that we did not say anything about the variance of returns. It is sufficient so to state that the three distributions we discussed (negatively skewed, positively skewed, symmetric) explicitly share the same mean but may also share the same variance.

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