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MANAGING CASH-IN RISK WITHIN PORTFOLIO INSURANCE STRATEGIES: A REVIEW

Abstract

A *portfolio insurance strategy* is a dynamic hedging process aiming to limit downside risk during a market downturn and allow investors to obtain equity market participation in the upside market. The biggest potential risk of implementing a portfolio insurance strategy is the so-called *cash-in risk*, i.e., the risk that the underlying asset registers huge drops before the portfolio can be rebalanced. In such cases, the value of the insured portfolio would fall below the floor (the insured capital), and the consequence is that the portfolio is fully monetized, not allowing the investor to recover the capital initially invested. First, this paper reviews the main properties of the most used allocation algorithm, the so-called *Constant Proportion Portfolio Insurance* (CPPI), and how the cash-in risk affecting this kind of allocation strategy can be modelled and hedged. Secondly, it describes the main extensions of CPPI proposed in the literature to improve its capability to reduce cash-in risk.

Keywords: OBPI, cash-in risk, CPPI, VPPI, EPPI

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1 Introduction

Portfolio insurance strategies were first introduced by Rubinstein and Leland (1976) after the collapse of stock markets (the *New York Stock Exchange's Dow Jones Industrial Average* and the *London Stock Exchange's FT 30*) which implied the pension funds withdrawal. In particular, the authors noted ex-post that the presence of an insurance could have convinced the investors not to leave the market, guarantying them the opportunity to take advantage of the rise of the same, an event that happened just a couple of years later. Perold and Sharpe (1988) state that portfolio insurance strategies can be classified into three categories: *option-based strategies*, *option-duplicating strategies* and *derivative-independent strategies*. The primary approach related to the first class of strategies is the so-called *option-Based Portfolio Insurance* (OBPI), which consists of buying a zero-coupon bond with maturity equal to the investment time horizon plus an option written on a risky asset. An option-duplicating strategy is an approach where the option is replicated with a self-financing strategy in order to overcome the lack of liquid options for long maturities. However, the low-interest rate levels which have characterized the markets in recent years are reducing the available risk budgets significantly. Such a market environment forces practitioners to rethink how to build their portfolios to simultaneously offer sustainable equity market participation and capital protection for the initial investment. In this direction, one choice is to consider dynamic risk management tools to protect portions of the initial investment by dynamically allocating wealth into risky and riskless assets. In this framework the *Constant Portfolio Portfolio Insurance* (CPPI) is one of the most used approaches. The CPPI method is obtained by rebalancing an initial portfolio at each observation time, evaluating the present value of the capital to be protected and then investing the available risk budget into risky assets while investing the remaining part of the portfolio in risk-free assets. Despite a significant simplicity and a remarkable ease of implementation, the CPPI strategy suffers a fundamental drawback represented by the risk that, after a severe market draw-down, the risk budget erases. This event is the so-called *cash-in risk* and it has mainly two consequences. Since after the cash-in event, the remaining portfolio is entirely invested into the riskless asset, the CPPI strategy might not guarantee *(i)* the capital initially invested at maturity and, *(i)* an equity market participation in case of subsequent rises of financial market. The paper is structured as follows: in Section 2, we recall the main concepts related to OBPI strategy; in Section 3, we provide an in-depth analysis of CPPI strategy by reviewing how cash-in risk can be modeled and hedged; in Section 4, we describe the main extensions of the CPPI strategy, i.e. the *Time Invariant Portfolio Protection* (TIPP) and the *Variable Proportion Portfolio Insurance* (VPPI), whose aim is to reduce the probability of cash-in event; in Section 5 we focus on an alternative way, more recently introduced, to hedge cash-in risk, based on the use of a particular kind of options; Section 6 concludes.

2 Option based Porfolio Insurance strategies

Option-based portfolio insurance (OBPI) is a methodology characterized by ensuring a minimal terminal portfolio value. Along the lines of Bertrand and Prigent (2005), it is possible to define the OBPI portfolio process $V^{OBPI} = \left\{V_t^{OBPI}\right\}_{t\in[0,T]}$, with initial value V_0^{OBPI}, as follows:

$$V_t^{OBPI} = qB_t + pC(t, S_t, K), \tag{1}$$

where $q \geq 0$ represents the number of the riskless asset purchased by the investor to protect the capital initially invested, $C(t, S_t, K)$ is the price of the call option, written on the risky asset S_t, having strike price K and maturity T and $p \geq 0$ is the number of call option that can be acquired at time $t = 0$, given the risk budget. The strategy is relatively simple to implement since it is static, i.e. no trading occurs in $[0, T]$. From eq. (1) it is straightforward to show that K represents the wealth the investor wishes to recover at maturity T. However, it may happen that European options, whose strike price is equal to the amount of wealth to be immunized, are not traded on the market. This implies that the investor must synthetically replicate the payoff at maturity of the option using a hedging strategy. Perfect hedging can be achieved only by assuming that the market is complete. However, it is well known that various sources of market incompleteness exist in terms of stochastic volatility and trading restrictions, making the contingent claim payoff unattainable. This implies that the standard OBPI approach is not always viable. This issue explains why dynamic Portfolio Insurance strategies such as CPPI, which will be examined in 3, have become prominent among practitioners.

3 The Constant Proportion Portfolio Insurance

The objective of the Constant Proportion Portfolio Insurance (CPPI) investors is twofold: participating in the upside potential of the risky reference portfolio, e.g. a market index, and at the same time, ensuring that the value of the portfolio at maturity V_T^{CPPI} is higher or equal to a guaranteed amount G ($V_{CPPI}^T \geq G$). The guarantee G is a proportion $PL \in (0,1]$ of the initially invested amount V_0^{CPPI}. These two goals are realized by dynamically allocating the initial wealth V_0^{CPPI} between a risk-free asset and a market index. In order to define the CPPI portfolio process, we begin by specifying the so-called *floor* F_t, representing the lowest acceptable value for the portfolio for each instant of time $t \in [0,t]$. The floor F_t is given by

$$F_t = F_T e^{-r(T-t)}, \tag{2}$$

where $F_T = G = PL \cdot V_0^{CPPI}$ and r is the risk-free interest rate. The next step is to compute the *cushion* C_t, which is the difference between the portfolio value V_t and the floor F_t. The exposure to the risky asset E_t is given by the product between the cushion C_t and the multiplier $m \in \mathbb{R}^+$. The latter parameter amplifies the risk budget and is exogenously set by the investor at $t=0$. Since the strategy is self-financing, the remaining part of the portfolio, i.e. $V_t^{CPPI} - E_t$, is invested into the risk-free asset. Then, the proportion of wealth invested to the risky asset for each instant of time $t \in [0,T]$ can be written as:

$$\alpha_t^{CPPI} = \min\left\{\frac{m(V_t^{CPPI} - F_t)^+}{V_t^{CPPI}}, LEV\right\}, \tag{3}$$

$$\beta_t^{CPPI} = 1 - \alpha_t^{CPPI}. \tag{4}$$

where $LEV \in (0,2]$ is the *maximum leverage factor*: to avoid excessive equity exposure, E_t is bounded to be at most $LEV \cdot V_t^{CPPI}$. We start by considering the case in which the CPPI portfolio is continuously rebalanced, meaning that the Exposure E_t and the investment in the riskless asset B_t are continuously adjusted. The main properties and the structure of the continuous-time CPPI allocation strategy, summarized 3.1, have been extensively studied in Black and Perold (1992).

3.1 CPPI with continuous rebalancing

In order to define the continuous-time CPPI portfolio process, we begin by specifing the so-called floor process $F = \{F_t\}_{t\in[0,T]}$ whose dynamic is given by

$$dF_t = rF_t dt, \tag{5}$$

with initial value $F_0 = V_0^{CPPI} \cdot PL \cdot e^{-rT}$. Then we define the process $V^{CPPI} = \left\{V_t^{CPPI}\right\}_{t\in[0,T]}$ with initial value V_0^{CPPI}, representing the portfolio value associated to CPPI strategy, namely:

$$V_t^{CPPI} = \alpha_t^{CPPI} S_t + \beta_t^{CPPI} B_t, \tag{6}$$

where α_t^{CPPI} (resp. β_t^{CPPI}) is depicted in eq. (3) (resp. eq. (4)). Furthermore, since the CPPI strategy is self-financing, the dynamics of V_t^{CPPI} is given by

$$dV_t^{CPPI} = \alpha_t^{CPPI} dS_t + \beta_t^{CPPI} dB_t. \tag{7}$$

Moreover, if we assume that the risky asset S_t follows a geometric Brownian motion, i.e.:

$$dS_t = \mu S_t dt + \sigma S_t dW_t^{\mathbb{P}}, \tag{8}$$

where $W = \{W_t\}_{t\in[0,T]}$ is a standard Brownian motion with respect to the real world measure $\mathbb{P}$, $\mu \in \mathbb{R}$ s.t. $\mu > r \geq 0$ and $\sigma \in \mathbb{R}_+$, Black and Perold (1992) explicitly derived the SDE satisfied by the cushion process given by $C = \{C_t\}_{t\in[0,T]}$, given by

$$\frac{dC_t}{C_t} = \Big(r + m(\mu - r)\Big)dt + m\sigma dW_t^{\mathbb{P}}. \tag{9}$$

From eq. (9), it is straightforward to show that the CPPI portfolio value is:

$$V_t^{CPPI} = (V_0 - G \cdot e^{-rT})e^{[r - m(r - \frac{\sigma^2}{2}) - \frac{m^2\sigma^2}{2}]t}\left(\frac{S_t}{S_0}\right)^m + G \cdot e^{-rT},\ t \in [0, T]. \tag{10}$$

Eq. (10) illustrates that within this framework, CPPI portfolio is equivalent to taking a long position in a zero-coupon bond with nominal value G in order to guarantee the capital at maturity, and investing the remaining part into a risky asset which has m times the excess return of S and is perfectly correlated with S. Moreover, it shows that the portfolio protection is efficient almost surely: the terminal value of the CPPI strategy is higher than the guarantee with probability one, regardeless of multiplier value. Indeed, the expected value of a CPPI-insured portfolio at maturity is equal to:

$$\mathbb{E}[V_T] = G + (V_0 - G \cdot e^{-rT})e^{[r+m(\mu-r)]T}. \tag{11}$$

which is always greater or equal with respect to the amount of capital that the investor wishes to recover at the end of the investment time horizon.

As highlighted in Balder et al. (2009); Cont and Tankov (2009), CPPI managers widely recognize the possibility of reaching the floor: there is a non-zero probability that, during a sudden downside movement of the underlying asset, the fund manager will not have time to readjust the portfolio, which crashes through the floor. This implies that the remaining portfolio value will be shifted entirely to the risk-free asset. Hence, it is no longer ensured that the strategy outperforms the prescribed floor. As mentioned in Section 1, this risk is known as cash-in risk.

Measuring the risk that the CPPI strategy is less than the floor is of practical importance for at least two reasons. Firstly, a CPPI strategy is combined with a guarantee for the investor: even if the floor has been broken, the CPPI issuer must pay the guaranteed amount F_T. Since the CPPI has fallen below the floor during the investment period, i.e. $V_T < F_T$, the issuer has to pay out more than the CPPI is worth. For this reason, an additional option can be added. Such an option is exercised if the value of the CPPI is below the floor. Secondly, CPPI strategies can be used to protect return guarantees embedded in unit-linked life insurance contracts. In the above case, maturity can be interpreted as retirement age, and guarantee as the amount which is at least needed by the insured.

The formal proof that there exist only two sources of cash-in risk is given in Schied (2013). The first source, extensively studied in Balder et al. (2009), is represented by discrete rebalancing of the CPPI portfolio. The second source of cash-in risk, modeled for the first time by Cont and Tankov (2009), is given from the fact that the price of the underlying risky asset may experience downward jumps.

3.2 CPPI with discrete time rebalancing

Let τ denote a sequence of equally spaced instants of time belonging to the interval $[0,T]$, i.e.:

$$\tau = \{t_0 = 0 < t_1 < \cdots < t_{n-1} = T\}, \tag{12}$$

where $t_{k+1} - t_k = \frac{T}{n}$ for $k = 0,\ldots,n-1$. We impose the rescrition that trading is only possible immediately after $t_k \in \tau$. This implies that the number of shares held in the risky asset is constant over the interval $(t_k, t_{k+1}]$ for $k = 0,\ldots,n-1$. However, the portion of CPPI portfolio invested in risky asset changes as risky asset price fluctuates. Thus, it is necessary to consider the number of shares held in the risky asset $\phi^{(S)}$ and the number of risk-free bond $\phi^{(B)}$. Along the lines of Balder et al. (2009), we indicate by $\phi_\tau = (\phi^{(S),\tau}, \phi^{(B),\tau})$ a discrete time CPPI if, for $t \in (t_k, t_{k+1}]$ and $k = 0,\ldots,n-1$, $\phi_t^{(S),\tau} := \max\left\{\frac{mC^\tau_{t_k}}{S_{t_k}}, 0\right\}$, where the cushion is given by

$$C^\tau_{t_{k+1}} = C^\tau_{t_0} \prod_{i=1}^{\min\{\nu,k+1\}} \left(m\frac{S_{t_i}}{S_{t_{i-1}}} - (m-1) \right), \tag{13}$$

with

$$\nu := \min\left\{t_k \in \tau | V^\tau_{t_k} - G \leq 0\right\}, \tag{14}$$

and $\nu = \infty$ if the minimum is not attained. Within this framework, the authors quantify the cash-in risk by computing the following quantities:

1. the *local shortfall probability*,

$$\mathbb{P}^{LSF} := \mathbb{P}(V^\tau_{t_{k+1}} \leq F_{t_{k+1}} | V^\tau_{t_k} > F_{t_k}) = \mathcal{N}(-d_2),$$

   where $d_2 = \frac{\ln\frac{m}{m-1} + (\mu - r)\frac{T}{n} - \frac{\sigma^2}{2}\frac{T}{n}}{\sigma\sqrt{\frac{T}{n}}}$,

2. the *shortfall probability*,

$$\mathbb{P}^{SF} := 1 - (1 - \mathbb{P}^{LSF})^n,$$

3. the *expected shortfall*,

$$ES := \mathbb{E}[G - V^\tau_T | V^\tau_T \leq G] = G + (V_0 - F_0)\left[E_1^n + e^{-r\frac{T}{n}} E_2 \frac{e^{rT} - E_1^n}{1 - E_1 e^{-r\frac{T}{n}}} \right],$$

   where

$$\begin{aligned} E_1 &:= me^{\mu\frac{T}{n}}\mathcal{N}(d_1) - e^{r\frac{T}{n}}(m-1)\mathcal{N}(d_2), \\ E_2 &:= e^{r\frac{T}{n}}[1 + m(e^{(\mu-r)\frac{T}{n}} - 1)] - E_1. \end{aligned}$$

   with $d_1 := d_2 + \sigma\sqrt{\frac{T}{n}}$.

Within this framework, two solutions have been proposed to ensure the effectiveness of the CPPI strategy. The first one is the following: given an estimate for μ and σ, it is possible to determine the value of the multiplier m and the number of rebalances n of $\phi^{(S),\tau}$ and $\phi^{(B),\tau}$ over $[0,T]$ such that the probability of falling below the guarantee G is bounded above a confidence level γ. The second one has been proposed by Bertrand and Prigent (2016). It is based on large deviation methods that the authors use to estimate the possible losses between two consecutive trading dates.

3.3 CPPI in presence of jumps in asset prices

As mentioned in Section 3.1, the second alternative to model cash-in risk is to allow for jumps in the risky asset dynamics without relaxing the continuous trading assumptions. CPPI strategies with the presence of jumps in stock prices were considered for the first time by Prigent and Tahar (2005) in a jump-diffusion model with finite intensity activity. However, the approach by Prigent and Tahar (2005) is to consider as a slight modification of the CPPI strategy, which incorporates an additional guarantee whenever the portfolio falls below the floor.

The first study devoted to quantifying cash-in risk within CPPI strategy by considering a more general framework including infinite activity jumps and stochastic volatility, is given by Cont and Tankov (2009). The reason behind introducing the above models is to highlight that cash-in risk cannot be attributed exclusively to trading restriction. This could give the wrong impression that this risk can be eliminated by more frequent rebalancing. Indeed, Cont and Tankov (2009) argues that by considering jumps in the risky asset price dynamic, there is a non-negligible residual cash-in risk for CPPI, even in continuous trading.

As in Section 3.1, assume a continuous time market model. Hence, we have to consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{0 \le t \le T}, \mathbb{P})$ with two $\mathbb{F}$-adapted processes describing the evolution for the riskless asset B_t, and the risky asset S_t. As in the previous Sections, we assume that r is constant over the investment time horizon $[0, T]$. Then, B_t has the following deterministic evolution:

$$dB_t = rB_t dt, \; t \in [0, T], \tag{15}$$

with $B_0 = b$. In this case we assume that the price process for the risky asset S_t is

$$dS_t = S_{t-} dZ_t, \tag{16}$$

where Z_t is a possible discontinous driving process, modeled as semimartingale. Moreover, in order to ensure the positivity of the price process, we assume that $\Delta Z_t > -1$. In a continuous-time setting the stopping time defined in eq. (14) becomes

$$\nu = \inf \{t \ge 0, V_t \le F_t\}. \tag{17}$$

If $t < \nu$, the CPPI strategy portfolio value satisfies

$$dV_t = m(V_{t-} - F_t)\frac{dS_t}{S_{t-}} + \Big(V_{t-} - m(V_{t-} - F_t)\Big)\frac{dB_t}{B_t}, \tag{18}$$

which implies the following dynamic for the cushion

$$\frac{dC_t}{C_{t-}} = m dZ_t + (1 - m) r dt. \tag{19}$$

Introducing the discounted cushion $C_t^* = \frac{C_t}{B_t}$ and applying Itô formula, we find

$$\frac{dC_t^*}{C_{t-}^*} = m(dZ_t - r dt). \tag{20}$$

Defining $L_t = L_0 + \int_0^t dZ_s - rt$, eq. (20) can be rewritten as $\frac{dC_t^*}{C_{t-}^*} = m dL_t$ or, equivalently, $C_t^* = C_0 \mathcal{E}(mL)_t$ where $\mathcal{E}$ denotes the stochastic (Doléans-Dade) exponential defined by

$$\frac{d\mathcal{E}(mL)_t}{\mathcal{E}(mL)_{t-}} = m dL_t. \tag{21}$$

If $\nu \geq t$, according to the definition of CPPI strategy, the value of the portfolio is entirely invested into the risk-free asset, in order to prevent further losses. This means that, when $\nu \geq t$, the value of C^*_t remains constant. Then, for any $t \in [0, T]$ we introduce a new process $\tilde{C}_t$ defined as the *stopped* process of C^* such that $\tilde{C}_t = C^*_{t\wedge\nu}$, where $t \wedge \nu := \min\{t, \nu\}$ and it can be explicitly written as:

$$\tilde{C}_t = C^*_0 \mathcal{E}(mL)_{t\wedge\nu}. \tag{22}$$

Hence, the CPPI portfolio may fall below the floor, even if the exposure and riskless asset investment are adjusted continuously. This is due to the fact that the stochastic exponential in eq. (22) can become negative in the presence of negative jumps of sufficient size of stock price.Within this framework, Cont and Tankov (2009) quantify in a meaningful way the cash-in risk by establishing a direct relationship between the value of the multiplier m and the risk of the insured portfolio. This allows choosing the multiplier based on the risk tolerance of the investor. In particular, the authors provide a Fourier transform method to compute the losses distribution and various risk measures (e.g., Value-at-Risk, expected loss, or the probability of loss) over a given time horizon. Moreover, they extend the framework described in this Section by adding stochastic volatility.

4 Some extensions of CPPI allocation strategy

In order to mitigate the cash-in risk which affects the CPPI strategy, several solutions have been proposed. In particular, the financial literature derived mainly two extensions of the CPPI allocation algorithm. The first modification, proposed by Estep and Kritzman (1988), is the so-called Time-Invariant Portfolio Protection (TIPP). It is based on the same rules as the CPPI allocation algorithm except for the floor computation. In fact, in this case, it is no longer linked to the risk-free interest rate r but to the portfolio's value. The second modification, proposed by Lee et al. (2008), is the so-called Variable Proportion Portfolio Insurance (VPPI), which concerns how the multiplier is set. In the standard CPPI, the multiplier m is fixed at $t = 0$, based on the investor's risk aversion, and remains constant throughout the investment time horizon $[0, T]$. Instead, the VPPI allows the multiplier to vary over time according to specific factors, such as the volatility of the risky asset underlying the strategy. In what follows, we will discuss how these new strategies affect the amount invested into risky assets over time and, consequently, their ability to guarantee at least G at maturity and equity market participation in the event of bull markets.

4.1 Time Invariant Portfolio Protection (TIPP)

Standard results about CPPI strategies are based on the assumption that the floor F_t evolves like the riskless asset B_t. However, this assumption is quite stringent since it makes the CPPI performances highly path dependent; any gains at a particular time $t \in [0, T]$ can be lost if the underlying asset price registers a considerable fall. To address this issue, Estep and Kritzman (1988) proposed the Time-Invariant Portfolio Protection (TIPP), i.e. a modified version of CPPI with a ratchet mechanism, able to lock in a proportion of the upside performance. In particular, this mechanism consists of a stochastic time-varying definition of the floor. The TIPP floor F_t is defined as the maximum between the usual CPPI floor and a percentage of the historical maximum portfolio value. The new floor $\tilde{F}_t$ satisfies

$$\tilde{F}_t = \max\left\{F_t, PL \cdot \sup_{s\leq t} V_s\right\}, \ \forall t \in [0, T], \tag{23}$$

where PL is the protection level. In this way, the floor jumps up with the portfolio value to reduce risky asset allocation when the market peaks. This new floor adjustment has some consequences on the allocation of the risky asset over time, especially in market scenarios where the risk-free interest rates are very low and sudden rises and falls of the risky asset might occur. Such a background sheds light on one of the main issues related to using CPPI: when the value of the risky asset increases, the CPPI portfolio value increases accordingly. If the level of the risk-free interest rate is low, then the growth rate of the CPPI portfolio will be higher than the corresponding growth rate of the floor. This implies that there is no longer significant portfolio protection after a very short period. Such a drawback is overcome by the TIPP allocation strategy, thanks to dynamics in eq. (23). Indeed, the growth rate of the TIPP floor is comparable to the portfolio for every $t \in [0, T]$, even if the market growth is sustained. Consequently, the TIPP exposure to the risky asset will generally be lower than the CPPI one. It will change more smoothly over time, furnishing better downside protection to the investor. However, in case of favourable market conditions, the TIPP strategy's overall return will be generally lower w.r.t. the standard CPPI one. This is in line with the empirical analysis carried out by Dichtl et al. (2017), in which the authors conclude that TIPP cannot be seen as an improved CPPI. Indeed, the former exhibits a significantly inferior performance for all applied performance measures, even if it can furnish better protection against cash-in risk.

4.2 Variable Proportion Portfolio Insurance (VPPI)

Instead of continuously rebalancing the floor, it is reasonable to make the multiplier dynamic. This change makes the CPPI a Variable Proportion Portfolio Insurance, the so-called VPPI. One of the main reasons to consider such an extension is to allow the investor to better adapt his portfolio strategy to market fluctuations. For example, suppose that for the first year of a five-year global management period, the forecast for the stock index's performance is that significant sudden drops can occur. In this case, at $t = 0$, the investor has to choose a relatively low multiplier. However, suppose substantial rises will occur in the future. In that case, the exposure corresponding to a small value of the multiplier initially chosen will not provide the opportunity to benefit from a bullish market. On the contrary, if the initial value of the multiplier is relatively high, sudden significant drops will imply that the portfolio may break the floor. This will imply that, at maturity, the investor might only recover G. The possibility of reducing the multiple during the investment period can prevent such an unfavourable event. This extension of the classical CPPI allocation strategy was introduced by Lee et al. (2008). In order to keep the simplicity of the model, the authors proposed a particular kind of VPPI strategy, the *Exponential Proportion Portfolio Insurance* (EPPI), where the multiplier changes according to the following criteria. In $t = 0$, the investor has to fix a reference price for the risky asset underlying the strategy. Such a reference price is the value of the risky asset before the portfolio rebalance, $S^{(0)}$. If the value of the risky asset after the portfolio rebalancing, $S^{(1)}$ is different from $S^{(0)}$, the multiplier changes according to the following rule:

$$m_t = \eta + \exp\left\{ a \ln\left(\frac{S^{(1)}}{S^{(0)}} \right) \right\},\ t \in [0, T], \tag{24}$$

where $\eta > 1$ is an arbitrary constant, and $e^{a \ln \frac{S^{(1)}}{S^{(0)}}}$ is the so called *dynamic multiple adjustment factor* (DMAF), with $a > 1$. The parameter a acts as the magnifier of the DMAF. More precisely, it is set greater than 1 for the enlargement effect on the number of holding shares when the stock price goes up, and the shrinkage effect on the number of holding shares when the stock price goes down w.r.t. the reference stock price $S^{(0)}$. Thus, including a DMAF into the multiplier could create a higher convex nature of the strategy, i.e. when the stock price goes up, the mechanism of the EPPI strategy creates more holding shares to perform an upside capture. In contrast, when the stock price goes down, the strategy creates fewer holding shares to provide a downside protection.

However, the EPPI strategy is not the only extension of the CPPI, which allows for a dynamic multiplier. Indeed, a wide range of models has been proposed directly linked to a risk management approach. In particular, the baseline assumption of these kinds of models is to fix the multiplier at each rebalancing date by considering a *local quantile guarantee condition* posed by

$$\mathbb{P}_{t_k}(C_{t_{k+1}} > 0 | C_{t_k} > 0) \geq 1 - \varepsilon, \tag{25}$$

where $\mathbb{P}_{t_k}(\cdot)$ denotes the conditional probability with respect to $\mathcal{F}_{t_k}$ and the parameter ε denotes an exogenously specified upper bound on the local shortfall probability. We further observe that

$$\mathbb{P}_{t_k}(C_{t_{k+1}} > 0 | C_{t_k} > 0) = \mathbb{P}_{t_k}\left(m_{t_k} \frac{S_{t_{k+1}}}{S_{t_k}} - (m_{t_k} - 1) > 0 \right). \tag{26}$$

Eq.(26) implies an upper bound $\bar{m}_{t_k}$ on the admissible multiplier, i.e. a gap control affords to limit the multiplier at t_k by $\bar{m}_{t_k}$. Denoting by $\tilde{F}_{t_k}$ the marginal distribution function of the standardized simple return $\frac{S_{t_{k+1}}}{S_{t_k}} - 1$, the upper bound is given by

$$\bar{m}_{t_k} = \left| E_{t_k}\left[\frac{S_{t_{k+1}}}{S_{t_k}} \right] - 1 + \sqrt{Var_{t_k}\left[\frac{S_{t_{k+1}}}{S_{t_k}} \right]} \tilde{F}_{t_k}^{-1}(\varepsilon) \right|^{-1}. \tag{27}$$

The condition in eq. (27) is used in Hamidi et al. (2009), who estimate the conditional upper multiplier by means of the *Value at Risk* (VaR). In particular, they resort to eight different methods of VaR calculations: one non-parametric method using the historical simulation approach; three parametric methods based on distributional assumptions: namely, the normal VaR, the RiskMetrics VaR based on the normal distribution, and the GARCH VaR based on the Student-t distribution; four semi-parametric methods using quantile regression to estimate the conditional autoregressive VaR (CAViaR): namely, the symmetric Absolute Value CAViaR, the Asymmetric Slope CAViaR, the IGARCH(1,1) CAViaR, and the Adaptive CAViaR. Afterwards, Hamidi et al. (2014) proposed a generalization of this class of models in which the conditional multiplier is based on a coherent risk measure, the expected shortfall. In this case, they estimate the conditional upper bound for the multiplier $\bar{m}_{t_k}$ using a Dynamic AutoRegressive Expectile (DARE) model.

5 Hedging cash-in risk through options

Another way to hedge cash-in risk within portfolio insurance strategies is to use options. Let us consider the Vanilla options written on the CPPI portfolio as an example. For instance, taking a long position on an at-the-money put option on the CPPI portfolio with a strike price at least equal to the minimum value that the investor requires at maturity is a natural way to hedge cash-in risk. Similarly, taking a long position on an at-the-money call option on the CPPI portfolio is a natural way to invest in a CPPI portfolio, preserving the capability to not pursue forward the investment in the case of closed out. The first option pricing model for the CPPI strategy was proposed by Escobar et al. (2011) and generalized by Wang and Tsoi (2013).

5.1 Pricing option on CPPI

As described in Section 3.2, a possible way to model cash-in risk is to allow trading of the CPPI portfolio only on specific dates. For this reason, Escobar et al. (2011), to develop a pricing formula for European options written on CPPI, decided to model the risky asset underlying the strategy using a geometric Brownian motion, restricting trading to discrete-time. The discrete-time process describing the evolution of the risky asset under the risk-neutral probability measure $\mathbb{Q}$ is:

$$\frac{S_{t_k}}{S_{t_{k-1}}} = \exp\left\{ \left(r - \frac{\sigma}{2} \right) \Delta t + \sigma W_{t_k} \right\}, \tag{28}$$

where $W_{t_k} \sim N(0, \Delta t)$ and τ, as in eq. (12), is a sequence of equally spaced instants of time of the interval $[0, T]$, i.e $\tau = \{t_0 = 0 < t_1 < \cdots < t_{N-1} < T_N = T\}$ and $\Delta t = \frac{T}{n}$ for $k = 1, \ldots, N-1$. Within this framework the value $V^\tau_{t_{k+1}}$ of the CPPI portfolio is:

$$V^\tau_{t_{k+1}} = e^{r(t_{k+1} - \min\{\nu, k+1\})}\left((V_0 - F_0)\prod_{i=1}^{\min\{\nu,k+1\}}\left(m\frac{S_{t_i}}{S_{t_{i-1}}} - (m-1)e^{r\Delta t}\right) + F_{t_{\min\{\nu,k+1\}}}\right), \tag{29}$$

where ν given in eq. (14) is the first time the portfolio value breaks through the floor. In order to compute the price of the European option, the authors first derived the price of discrete time CPPI by making the following assumption. Since the CPPI strategy is by definition self-financing, then under the risk-neutral probability measure the expected terminal portfolio value is $\mathbb{E}^{\mathbb{Q}}[V_T] = e^{rT}V_0$. However, for the investor, the value of the CPPI strategy is not equal to V_0. As pointed out in Section 3.1, the CPPI is combined with a guarantee for the investor; this means that the issuer of the CPPI should guarantee a payoff equal to or greater than the floor and therefore has to carry the cash-in risk. Consequently, the payoff at maturity of the discrete CPPI is:

$$CPPI_T = \max\{V_T, F_T\}. \tag{30}$$

Then, the expected value at maturity T, under the risk-neutral probability measure $\mathbb{Q}$ can be written as:

$$\mathbb{E}^{\mathbb{Q}}[CPPI] = \mathbb{E}^{\mathbb{Q}}[CPPI_T|C_1] + \mathbb{E}^{\mathbb{Q}}[CPPI_T|C_2], \tag{31}$$

where $\mathbb{E}^{\mathbb{Q}}[CPPI_T|C_1]$ (resp. $\mathbb{E}^{\mathbb{Q}}[CPPI_T|C_2]$) is the expected value of the portfolio at maturity when the portfolio does not fall below the floor (resp. the expected value of the portfolio when the CPPI has fallen below the floor) over $[0, T]$. When the underlying risky asset follows the process depicted in eq. (28), both of them can be evaluated in closed form. Indeed, the authors proved that:

$$\mathbb{E}^{\mathbb{Q}}[CPPI_T|C_1] = F_0\mathcal{N}(s_2)^N + (V_0 - F_0)[m\mathcal{N}(s_1) - (m-1)\mathcal{N}(s_2)]^N, \tag{32}$$

$$\mathbb{E}^{\mathbb{Q}}[CPPI_T|C_2] = F_0(1 - \mathcal{N}(s_2)^N), \tag{33}$$

where $s_1 = \frac{\ln m - \ln(m-1) + \frac{\sigma^2}{2}\Delta t}{\sigma\sqrt{\Delta t}}$ and $s_2 = s_1 - \sigma\Delta t$. In order to compute the price of the European option on discrete CPPI, the authors distinguish two cases. The first case is when the strike price, K, is equal to the value of the guarantee at maturity F_T. In this case, the option on CPPI ends up in the money at maturity T if the strategy has not defaulted until maturity, i.e. $C^\tau_{t_{k+1}} > 0, k = 1, \ldots, N-1$. Then, a portfolio composed by the option on a CPPI with $K = F_T$ and a zero coupon bond with nominal value K, is exactly equal to a CPPI with floor F. For this reason when $K = F_T$, the price of the call option on CPPI with discrete rebalancing is exacty equal to the difference between the expected value of discrete CPPI in $t = 0$ given by eq. (32) and F_0:

$$C_t = (V_0 - F_0)[m\mathcal{N}(s_1) - (m-1)\mathcal{N}(s_2)]^N. \tag{34}$$

The second case is when the strike price is greater than the guarantee amount at maturity ($K \geq F_T$). In this more general case the option ends up in the money at maturity T if the CPPI has not fallen below the floor in t_k, $k = 1, \ldots, N-1$, and if the value V_T is greater than the strike price K. For this reason the option pricing formula defined in eq. (34) becomes:

$$C_t = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\left(F_T - K + (V_0 - F_0)\prod_{i=1}^{N}(me^{(r-\frac{\sigma^2}{2})\Delta t + \sigma W_{t_i}} - (m-1)e^{r\Delta t})\right)^+ \prod_{i=1}^{N-1}\mathbb{1}_{\{V_{t_l} > F_{t_l}\}}\right]. \tag{35}$$

In this more general case, the option price on discrete CPPI and very useful sensitivities like Delta, Gamma, Vega, Rho and Theta can be obtained using Monte Carlo simulations in eq. (35).

As explained in Section 3, the second source of cash-in risk for the CPPI strategy can be modelled by adding jumps into the dynamic of the underlying risky asset. Because of this, an alternative way to price the European option on CPPI is to consider a continuous time setting and a jump-diffusion model for the dynamics of the underlying risky asset. Wang and Tsoi (2013) provided the first work in this direction. In particular, they developed a semi-closed formula to price European options written on the CPPI strategy when the underlying risky asset evolves according to the model of Merton (1976). Moreover, since the market is not complete in a jump-diffusion setting, the payoff of these particular contingent claims is no longer attainable. For this reason, the authors developed a particular hedging strategy for this model, the so-called mean-variance hedging.

5.2 Structured product written on a modified version of the CPPI strategy

Using options linked to portfolio insurance strategies can be considered a suitable way to obtain downside protection when the underlying risky asset has experienced heavy losses during the investment time horizon. However, this method cannot offer equity market participation if the risky asset recovers nicely after a severe market draw-down. Di Persio et al. (2021) introduced a modified version of the standard CPPI by defining a minimum threshold always invested in the risky asset to overcome the latter scenario. They called this new kind of strategy CPPI with *guaranteed minimum equity exposure* (GMEE-CPPI). In particular, they introduced the GMEE α_{min} with $0 \leq \alpha_{min} \leq 1$ in eq. (3) obtaining

$$\alpha_t := \max\left\{\min\left\{\frac{m(V_t^{CPPI} - F_t)}{V_t^{CPPI}}, LEV\right\}, \alpha_{min}\right\},\ t \in [0,T]. \tag{36}$$

Thanks to eq. (36), the equity participation will never go below α_{min} even in case of a severe market drop. However, at the same time, it would mean that this adjusted CPPI allocation implemented in a real portfolio might not be able to protect the invested capital. For this reason, the authors introduced a new structured product consisting in a combination of a CPPI strategy and an OBPI one. In particular, this new strategy can be summarized into the following key points:

(i) a significant proportion of the initial portfolio value is invested in time-congruent zero coupon bonds following the classical OBPI approach described in Section 2,

(ii) the remaining part of the portfolio is put into an exotic call option linked to a GMEE-CPPI strategy where the CPPI portfolio has an equity index as a risky asset.

In particular, the authors provided historical simulations showing how the risk-return profile changes according to the market environment and describing the option price behaviours under different frameworks, namely, when the underlying is a pure risky asset, a CPPI strategy, or a CPPI–GMEE based one.

The authors argued that this new method provides a valuable compromise between a pure risky asset investment strategy and a traditional CPPI one. Indeed, this innovative method overcame, at the same time, different problems since the use of the OBPI can drastically reduce the cash-in risk, and the use of GMEE-CPPI as the underlying risky asset can ensure some equity market participation.

6 Conclusion and further research

In the present paper, we have reviewed the main properties of Portfolio Insurance strategies and the main risk affecting these particular kinds of dynamic hedging processes, the so-called cash-in risk. We have focused on modelling and hedging cash-in risk within the prominent portfolio insurance strategy, the Constant Proportion Portfolio Insurance (CPPI). First, we have introduced the main properties of the CPPI strategy within three different frameworks. In particular, we consider the CPPI strategy when the underlying risky asset follows *(i)* a geometric Brownian motion, *(ii)* a more general jump-diffusion process, and *(iii)* a geometric Brownian motion where trading is restricted to discrete time. We provide an in-depth analysis of how cash-in risk can be modelled and hedged for each setting mentioned above. Then, we analyzed the most important modifications of the CPPI strategy designed to reduce the probability of cash-in risk. The first is obtained by considering a stochastic time-varying definition of the floor process, the so-called Time-Invariant Portfolio Protection (TIPP). The second one is obtained by varying the multiplier over time according to market fluctuations, the so-called Variable Proportion Portfolio Insurance (VPPI). Lastly, we have reviewed another way to hedge cash-in risk using options linked to the CPPI strategy. Within this framework, we analyze a two-step principal protection strategy obtained by combining a modification of the CPPI, the so-called CPPI with guaranteed minimum equity exposure (GMEE-CPPI), and a classical OBPI strategy. As discussed in Section 5.2, this novel approach, introduced by Di Persio et al. (2021), represents a concrete innovation in the literature related to portfolio protection strategies. Along this line of research, further contributions could be made by including more structured derivatives evaluated concerning general stochastic volatility models with the presence of jumps. Moreover, other inputs for further research can be represented by furnishing a sensitivity analysis of the CPPI-GMEE approach w.r.t. to changes in market parameters. Lastly, another line of research can be represented by comparing options on CPPI with options on other dynamic asset allocation strategies, such as the VolTarget ones, allowing the CPPI-GMEE to have lock-in elements.

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