Annali del Dipartimento di Metodi e Modelli per l'Economia il Territorio e la Finanza
2022, DOI: 10.13133/2611-6634/1415

E-ISSN 2611-6634

ISSN 2385-0825

Research article

First published online: December 31, 2022

Ilaria Stefani*

AN OVERVIEW ON DYNAMIC PORTFOLIO ALLOCATION

Abstract

A portfolio allocation problem relies upon the decision process to establish how resources must be allocated among different possible investments. Investors are interested in gaining as much as possible from their investment, but at the same time, they are concerned with the risks they have to face. Investors aim to maximize their returns without exceeding a certain level of risk. Moreover, this behavior has to be mathematically modeled, resorting to the optimal control theory and the maximization of expected utility. This paper reviews the literature on portfolio allocation, to give a complete picture of what has been done, as well as, possible contributions for future research.

Keywords: asset allocation, stochastic volatility, co-jumps, recursive preferences, dynamic programming.

*Department of Methods and Models for Territory, Economics and Finance, Sapienza University of Rome, Italy.

1 Introduction

The first attempt to solve a portfolio allocation problem is due to Markowitz (1952), with the celebrated *mean-variance approach*. The latter continues to be widely applied in several financial frameworks, such as risk management, mainly because of its simplicity, since only the knowledge of the expectation and covariances of random variables is required.
The main drawback of this approach is the static nature of the optimal allocation. Indeed the initial wealth is allocated between different assets at the beginning of the time horizon without allowing for changes in the allocation until maturity.
Moreover, this behavior ignores the presence of price volatility, so adopting this type of allocation model could result in large losses.

A solution widely adopted in the literature is to consider continuous-time models for price dynamics: by allowing continuous trading, the investor can immediately react to possible changes in price volatility.

In this view, the pioneering work of Merton (1971) can be considered a starting point for continuous-time portfolio management. In this setting, the asset allocation problem is solved using the stochastic control method, and the optimal portfolio rules are expressed in terms of solutions to the second-order partial differential equation (the so-called *Hamilton-Jacobi-Bellman* (HJB) equation). In this seminal paper, an investor willing to allocate his wealth between stock and a risky asset is taken into account.

The bond price grows at the constant interest rate r, and the stock price dynamic follows the Black-Scholes model

$$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t,$$

where $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$ and Z is a standard Brownian motion. We assume that the investor's wealth $X = \{X_t\}_{t\in[0,T]}$ is apportioned among the risk-free asset and the stock, so that, for any $t \in [0, T]$, the non-negative process X is governed by the following diffusion dynamics:

$$dX_t = \alpha_t X_t((\mu - r)dt + r)dt + \alpha\sigma X_t dZ_t, \tag{1}$$

where α represents the proportion of wealth invested at time t into the stock and observes that the proportion invested into the riskless asset is equal to $1 - \alpha$.
The objective function to maximize over $\alpha \in A$ is of the form

$$U(X_T) = \mathbb{E}\left[\int_0^T f(X_t, \alpha)dt + g(X_T)\right], \tag{2}$$

where f and g are measurable functions on $\mathbb{R}^d \times A$, satisfying suitable integrability conditions and ensuring that the expectation in (2) is well-defined. The dynamic programming method to solve (2) consists in defining the value function, that corresponds to the maximum value for (2) when varying initial states. In symbols, we have

$$v(t, x) = \sup_{\alpha\in A} \mathbb{E}[U(X_T^{t,x})], \quad (t, x) \in [0, T] \times \mathbb{R}^d, \tag{3}$$

where U is an increasing and concave function on $\mathbb{R}^d$.

The corresponding HJB equation for (3) is

$$\frac{\partial v}{\partial t} + \sup_{\alpha\in\mathbb{R}}\left[(\alpha(\mu - r) + r)x\frac{\partial v}{\partial x}(t, x) + \frac{1}{2}\alpha^2\sigma^2x^2\frac{\partial^2}{\partial x^2}(t, x)\right] = 0, \tag{4}$$

for $(t, x) \in [0, T] \times \mathbb{R}^d$, with terminal condition

$$v(T, x) = U(x), x \in \mathbb{R}^d.$$

Proceeding with the first-order condition for (4), the candidate for the optimal allocation is obtained as follows

$$\hat{\alpha}(t, x) = -\frac{\mu - r}{\sigma^2} \frac{\frac{\partial v}{\partial x}}{x \frac{\partial^2 v}{\partial x^2}}(t, x). \tag{5}$$

Merton has also shown that Equation (4) can be explicitly solved for the special case of the utility function, as when we choose a CRRA (*Constant Relative Risk Aversion*) utility function,

$$\begin{cases} \frac{1}{1-\gamma} x^{1-\gamma},\ if\ x \geq 0 \\ -\infty,\ if\ x \leq 0 \end{cases}$$

where γ is the coefficient of risk aversion.

It is worth pointing out that the formal derivation of the HJB equation is justified by a verification theorem, which states that, when there exists a smooth solution to the HJB, such a solution coincides with the value function, see Pharm (2007).

An alternative approach to solving optimal allocation problems is the so-called *martingale method* developed by Plisca (1986), Karatzas et al. (1987) and Cox and Huang (1989), such an alternative is based on the transformation of the optimal portfolio allocation problem into a static optimization problem with the determination of the optimal terminal wealth, finding a portfolio strategy that leads to optimal terminal wealth. Several authors extend this approach, we recall that Carr et al. (2001) obtained optimal consumption and investment plans in a complete market when the underlying asset price is a pure jump Lévy process. Schroder and Skiadas (1999) developed the utility gradient approach for computing portfolios and consumption plans that maximize *Stochastic differential utility* function (SDU) and prove the existence, uniqueness, and basic properties of a parametric class of homothetic SDUs. A few years later Schroder and Skiadas (2003) derived closed-form solutions for the optimal consumption and trading strategy in terms of the solution to a single constrained BSDE using the utility gradient approach methodologically. The focus of this paper is the necessary and sufficient first-order conditions of optimality that one would be to solve to compute a solution.

2 Investments under Uncertainty

In a context of uncertainty, an investor chooses among different strategies based on his/her preferences, and this is done in terms of *utility function,* according to the *expected utility criterion*, see e.g. Von Neuman and Morgenstern (1947).

If we assume that an investor compares random returns whose probability distributions are known on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we denote by $\succ$ the *preference order* that satisfies the *Von-Neumann Morgenstern criterion,* so that

$$X_1 \succ X_2 \iff \mathbb{E}[U(X_1)] > \mathbb{E}[U(X_2)], \tag{6}$$

where X_1, X_2 are random variables and U is an increasing real-valued function. The choice of the utility function allows defining the concept of *risk aversion* and *risk premium* of the investor. Among the properties that the utility function must have we recall the *concavity* of the utility function. If we have a utility function that satisfies *Jensen's inequality*, then an agent prefers to get the expectation $\mathbb{E}[X]$ of this return (with certainty), that is:

$$U(\mathbb{E}[X]) \geq \mathbb{E}[U(X)], \tag{7}$$

which holds only for concave functions. For a risk-averse agent with concave utility function U we can define the risk premium associated with a random return X he is willing to pay, to get a certain gain:

$$U(\mathbb{E}[X] - \pi) = \mathbb{E}[U(X)]. \tag{8}$$

The quantity $(\mathbb{E}[X]-\pi)$ is called *certainty equivalent* of X, which is smaller than the expectation of X.

It is easy to obtain the *absolute risk aversion* at level x, given by:

$$\alpha(x) = -\frac{U''(x)}{U'(x)}, \tag{9}$$

that is the *Arrow-Pratt* coefficient of absolute risk aversion of U at level x.

In the context of dynamic programming, the choice of the utility function is very relevant. The most preferred class of utility functions is given by HARA (*Hyperbolic Absolute Risk Aversion*) functions, where the inverse of absolute risk aversion is linear in consumption. We mention the quadratic, exponential, and isoelastic preferences. The most popular preference specification is either isoelastic preferences CRRA, or power utility see Hansen and Singleton (1982). The power utility has the following specification:

$$U(x) = \begin{cases} \frac{1}{1-\gamma}x^{1-\gamma}, \; if \; x > 0 \\ -\infty, \qquad if \; x \leq 0 \end{cases}, \tag{10}$$

where $\gamma > 0$ and the second part of the utility specification impose a non-negative wealth constraint.

The parameter γ plays a crucial role, indeed it is the risk aversion parameter and summarizes the behavior of the consumer toward risk, but at the same time its reciprocal is equal to the *constant elasticity of intertemporal substitution* (EIS) in consumption, and measures how consumption changes in time. So, only one parameter governs both investor risk aversion and EIS, and the latter is independent of the level of consumption.

To give rational support to observable investors' behavior decision-makers might take into account more complex utility functions, to use a nonlinear aggregator to bring together present and future utility thus dropping the hypothesis of expected utility.

In this perspective, the literature refers to the so-called *stochastic differential utilities* (SDU) introduced by Epstein and Zin (1989), who derive a parametrization of recursive utility in a discrete-time setting, and generalized by Duffie and Epstein (1992) and Fisher and Gilles (1999) in a continuous-time setting.

The Duffie and Epstein (1992) parametrization is

$$J = \mathbb{E}_t \left[\int_t^\infty f(C_s, J_s) ds \right], \tag{11}$$

where $f(C_s, J_s)$ is a normalized aggregator of current consumption and utility with the following form:

$$f(C, J) = \frac{\beta}{1 - \frac{1}{\psi}}(1-\gamma)J \left[\left(\frac{C}{((1-\gamma)J)^{\frac{1}{1-\gamma}}} \right)^{1-\frac{1}{\psi}} - 1 \right], \tag{12}$$

where $\beta > 0$ is the *rate of time preferences*, $\gamma > 0$ is the *coefficient of relative risk-aversion* and $\psi > 0$ is the *elasticity of intertemporal substitution* in consumption. In this setting there is a separation among preferences parameters, so the degree of risk aversion γ is disentangled from the elasticity of intertemporal substitution of consumption EIS. The Epstein-Zin preferences can be seen as a generalization of the standard time additive expected utility function, in that the former collapse to the power utility when the elasticity of intertemporal substitution of consumption EIS equals the reciprocal of the coefficient of relative risk aversion, so when we set $\psi = \frac{1}{\gamma}$.

3 Models for optimal portfolio allocation

3.1 Power Utility

If we focus on portfolio decisions, the literature has produced numerous contributions especially when the investor's decisions are based on his risk aversion, through the isoelastic utility function. Merton (1971) firstly studied optimal portfolio allocation, and derived optimal consumption and investment rules maximizing the expected utility in an economy composed of a riskless asset and a risky stock, when the asset price follows a Geometric Brownian motion. He showed that when investors have some special case of the HARA utility function, the maximization problem can be solved in a closed form.

Subsequently, several authors generalized Merton's work in incomplete markets: we recall, among others, the work of Liu et al. (2003), who used the event-risk framework of Duffie et al. (2000) to provide analytical solutions to optimal portfolio problems by assuming discontinuities in the state variable dynamics. In particular, Liu et al. (2003) provided two examples to illustrate their results. In the first one, they consider a model where the risky asset follows a jump-diffusion process with deterministic jump size, assuming constant volatility. They find that an investor would hold less of the risky asset when the price jumps occur. In the second example the authors consider a model where both the risky asset $S = \{S_t\}_{t\in[0,T]}$ and its return volatility $V = \{V_t\}_{t\in[0,T]}$ follow jump-diffusion processes with deterministic jump sizes, in particular, consider the following dynamics:

$$\begin{cases} dS_t = (r + \eta V_t - J\lambda V_t)S_t dt + \sqrt{V_t} S_t dZ_t^{(1)} + JS_{t^-} dN_t \\ dV_t = (k\theta - kV_t - \xi\lambda V_t)dt + \sigma\sqrt{V_t})dZ_t^{(2)} + \xi dN_t \end{cases}, t \in [0, T] \tag{13}$$

where $Z_t^{(1)}$and $Z_t^{(2)}$ are standard Brownian motions with correlation ρ. V_t is the instantaneous variance of diffusive returns and N_t is a Poisson process with stochastic arrival intensity λV_t, assuming λ to be nonnegative. The parameter $\theta > 0$ is the long-run mean, $k > 0$ is the mean-reversion rate, $\sigma > 0$ is the vol-of-vol coefficient and η is the market risk premium. Finally, $J, \xi > 0$ are the jump sizes, assumed to be constant.
Defining the indirect utility function W as

$$W(X, V, t) = \max_{\phi_s, t\leq s\leq T} \mathbb{E}_t \left[U(X_T)\right], \tag{14}$$

we have the following HJB equation for the indirect utility function W:

$$\max_{\phi}\Bigg(\frac{\phi^2 X^2 V}{2} W_{XX} + \phi\rho\sigma XVW_{XV} + \frac{\sigma^2 V}{2} W_{VV} + (r + \phi(\eta - J\lambda)V)XW_X + \\ (k\theta - kV - \xi\lambda V)W_V + \lambda V(\mathbb{E}[W(X(1 + \phi J), V + \xi, t)] - W) + W_t\Bigg) = 0 \tag{15}$$

where W_X, W_V and W_t denote the derivatives of $W(X, V, t)$ with respect to X, V and t and similarly for the higher derivatives. Proceeding with a *Guess and Verify* approach, the problem is solved assuming that the indirect utility function is of the form

$$W(X, V, t) = \frac{1}{1-\gamma} X^{1-\gamma} \exp\{F_t V_t + G_t\}, \tag{16}$$

where F_t and G_t are deterministic functions of time .

Depending on the model F_t and G_t can be solved in closed form or numerically, the latter case is that of this model. This model led to the following expression for optimal portfolio weights:

$$\phi^* = \frac{\eta - J\lambda}{\gamma} + \frac{\rho\sigma F_t}{\gamma} + \frac{\lambda J}{\gamma}(1 + J\phi^*)^{-\gamma} e^{\xi F_t}, \tag{17}$$

where F is the solution to the following ordinary differential equation

$$\begin{aligned} &\dot{F}_t + \frac{\sigma^2 F_t^2}{2} + (\phi^* \rho\sigma(1-\gamma) - k - \xi\lambda)F_t + \\ &\left(\frac{\gamma(\gamma-1)\phi^{*2}}{2} + (\eta - J\lambda)(1-\gamma)\phi^* + \lambda M_2 - \lambda\right) = 0, \end{aligned} \tag{18}$$

with $\dot{F}_t = \frac{\partial F}{\partial t}$, Liu et al. (2003) for further technical details.
It is possible to note that the investment horizon affects optimal portfolio weights through the hedging component of the demand for the risky asset ($\frac{\rho\sigma F_t}{\gamma}$) and from the static hedging component ($e^{\xi F_t}$) in Equation (18). Moreover, the authors found that volatility jumps can have a significant effect on optimal portfolio: in presence of jumps in volatility, the investor increases the optimal allocation in the risky asset. This means that volatility jumps have a compensating effect concerning price jumps.

Liu (2007) solved dynamic portfolio choice problems using affine models to face stochastic volatility. Three applications are presented: the first one is the bond portfolio selection problem when bond returns are described by quadratic term structure models; the second one is the stock portfolio selection problem when volatility is stochastic as in the Heston model; the last application is a portfolio selection problem in the incomplete market when the interest rate is stochastic and stock returns have stochastic volatility.
Very relevant to understand the subsequent works is to focus on the second application, the following dynamics are considered:

$$\begin{cases} dS_t = (r + \eta V_t)S_t dt + \sqrt{V_t} S_t dZ_t^{(1)} \\ dV_t = (k\theta - kV_t)dt + \sigma\sqrt{V_t} dBZ_t^{(2)} \end{cases}, t \in [0, T] \tag{19}$$

where $S = \{S_t\}_{t\in[0,T]}$ is the stock price, $V = \{V_t\}_{t\in[0,T]}$ is the volatility and $\eta V_t = \mu - r$ is the risk premium.
Associated with the system (19) we have the following HJB equation:

$$\begin{aligned} &\max_{\phi}\left(\frac{\phi^2 X^2 V}{2} W_{XX} + \phi\rho\sigma XVW_{XV} + \frac{\sigma^2 V}{2} W_{VV} + (r + \phi\eta VXW_X + \right. \\ &\left. (k\theta - kV)W_V + W_t\right) = 0, \end{aligned} \tag{20}$$

where $W_X = \frac{\partial W}{\partial X}$, $W_V = \frac{\partial W}{\partial V}$, $W_t = \frac{\partial W}{\partial t}$, $W_{XX} = \frac{\partial^2 W}{\partial X^2}$, $W_{VV} = \frac{\partial^2 W}{\partial V^2}$ and $W_{XV} = \frac{\partial W}{\partial X \partial V}$.
The problem is solved assuming Equation (16) as an indirect utility function.
The optimal stock portfolio weight ϕ_s^* is given by

$$\phi_s^* = \frac{1}{\gamma}\eta + \rho\sigma F_t, \tag{21}$$

where

$$F_t = \begin{cases} -\frac{2[exp(k_2 t)-1]}{(k_1+k_2)[exp(k_2 t)-1]+2k_2}\delta_v \quad if \quad k_2^2 \geq 0 \\ -\frac{2}{k_1 + \zeta\frac{cos(\zeta t/2)}{sin(\zeta t/2)}}\delta \quad if \quad \zeta^2 \geq 0 \end{cases}, \tag{22}$$

and

$$k_1 = k - \frac{1-\gamma}{\gamma}\alpha\sigma\rho, k_2 = \sqrt{k_1^2 + 2\delta[\rho^2 + \gamma(1-\rho^2)]\sigma^2}\delta = -\frac{1-\gamma}{2\gamma^2}\alpha^2, \zeta = -ik_2.$$

The stock portfolio weight in the stochastic volatility model is a nonmonotonic function of risk aversion. Indeed the myopic component decreases when risk aversion increases. A surprising and criticized feature of the optimal portfolio weight is that it is independent of the variance V. One might be expected the agent to hold more stocks when the volatility is low and less when the volatility is high, but this is true if the risk premium is independent of the volatility: in this model, the risk premium is proportional to the conditional variance. Hence, when the variance is high, the risk premium is also high, and vice versa.

We observe that the aforementioned findings are not related to a specific verification theorem, so we do not know under which conditions the candidate for the optimal portfolio strategy is the unique optimal solution to the allocation problem.

The work by Kraft (2005) filled this gap, by proving a verification result within the stochastic volatility framework, the author also derived the parameters' conditions ensuring well-defined candidates for the solution of the problem, a condition that ensures that these are indeed the solutions of the optimal portfolio process given the value function.

In contrast, Pharm (2022) and Fleming and Hernandez-Hernandez (2003) derived explicit verifications results proving that their portfolio strategies are indeed optimal. In particular, Pharm (2022) considered a multidimensional model for securities with stochastic volatilities, assuming certain Lipschitz conditions for coefficients. Such a choice excluded *de facto* the Heston model.

On the other hand, Fleming and Hernandez-Hernandez (2003) assumed the asset volatility to be a function σ depending on a state process with constant volatility, with boundedness assumptions for σ and σ'. Also, in this case, the Heston model is not included.

Buraschi et al. (2010) developed a new framework for multivariate intertemporal portfolio choice, in which the correlations across asset classes and volatilities are stochastic. In this setting, volatilities and correlations are conditionally correlated with returns. To model stochastic variance-covariance risk, the authors specified the covariance matrix process as a Wishart diffusion process. More precisely, we refer to the bi-dimensional case, so that the dynamics of the price vector $S_t = (S_1, S_2)^T$ are described by the bivariate stochastic differential equation:

$$\begin{cases} dS_t = diag(S_t)\left[(r + \mathbf{\Lambda}(\Sigma,t))dt + \Sigma_t^{1/2} dZ_t^{(1)}\right] \\ d\Sigma_t = [\Omega\Omega' + K\Sigma_t + \Sigma_t K']\,dt + \Sigma_t^{1/2} dZ(t)^{(2)} Q + Q'(dZ_t^2)'\Sigma_t^{1/2} \end{cases}, t \in [0,T] \tag{23}$$

where $diag(S_t)$ is the square matrix with S_t in the diagonal and 0 on the off-diagonal elements, $\mathbf{\Lambda}(\Sigma,t))$ is a vector of possibly state-dependent risk premia. Here the latter is assumed to be a constant market price of variance-covariance risk, namely $\mathbf{\Lambda}(\Sigma,t)) = \Sigma(t)\eta$ with $\eta = (\eta_1,\eta_2)^T \in \mathbb{R}^2$. Finally, the processes $Z_t^{(1)} \in \mathbb{R}^{2\times 1}$, and $Z_t^{(2)} \in \mathbb{R}^{2\times 2}$ are matrix Wiener processes, while K, Q, Ω are 2×2 square matrix (with Ω invertible). One property that the model must satisfy is $\Omega\Omega' = \varphi QQ'$ with $\varphi \in \mathbb{R}$ and $\varphi > N-1$.
The Wiener processes determining shocks in prices S_t and in the variance-covariance matrix Σ_t are correlated according to

$$Z_t^{(1)} = Z_t^{(2)}\rho + \sqrt{1-\rho'\rho}\,Z_t^{(3)}, \tag{24}$$

where $\rho = (\rho_1, \rho_2)$ is a vector of correlation parameters, and $Z_t^{(3)}$ is a two-dimensional standard Brownian motion, independent of $Z_t^{(2)}$.
The optimal weights obtained in such a framework are:

$$\phi^* = \frac{\alpha}{\gamma} + 2F_t Q'\rho, \tag{25}$$

with F_t being the solution to the following equation

$$\dot{F}_t + \Gamma' F_t + F_t \Gamma + 2F_t \Lambda F_t + C = 0, F_T = 0. \tag{26}$$

The optimal weight in (25) is the sum of a *myopic demand* and *hedging demand* of the risky asset, as already highlighted in Liu (2007) in the univariate case, and it is independent of the variance. Moreover, the authors pointed out that the hedging demand is typically larger than in univariate models, and it includes a significant covariance hedging component.

In a similar context, Jin and Zhang (2012) examined a multi-asset model with constant volatility when the price can jump. Differently from other authors, such as Das and Uppal (2004) and Ait-Sahalia et al. (2009), who focused on one type of jump and assumed constant coefficients in their dynamics, Jin and Zhang (2012) includes multiple types of jumps and a large number of assets and state variables. Furthermore, this paper also incorporates model uncertainty into the portfolio choice problem, tackling the problem by focusing on ambiguity aversion to inaccurate estimates of parameters associated with jumps. The theoretical results showed that ambiguity aversion increases the effect of jumps for a risk-averse investor, thus the exposure to jumps is smaller in the case of an investor not neutral to ambiguity.

The evidence highlighted in the works mentioned above shows once again that the optimal weights result to be independent of the instantaneous volatility when the dynamics of the risky asset are affine, implying a static allocation mechanism.

To overcome this drawback, Oliva and Renò (2018) study a continuous time optimal portfolio allocation with volatility and co-jump risk in a multi-asset framework. They deviate from affine models by specifying a Wishart jump-diffusion for the co-precision (the inverse of the covariance matrix), generalizing the dynamics in Chacko and Viceira (2005) to a multivariate setting. They solved the optimal portfolio problem, providing an exact solution to this problem in the absence of jumps and an approximated solution in the presence of co-jumps. More precisely, it is assumed there exist N risky assets, whose prices are given by $S_t = (S_{t,1}, \cdots, S_{t,N})^T \in \mathbb{R}^{\mathbb{N} \times 1}$ with dynamics

$$\begin{cases} dS_t = diag(S_t)\left[(\eta dt + \sqrt{Y_t^{-1}} dZ_t^{(1)} + J dN(\lambda)_t\right] \\ dY_t = \left[\Omega\Omega' + KY_t + Y_t K'\right] dt + \sqrt{Y_t} dZ_t^{(2)} Q + Q'(dZ_t^2)' \sqrt{Y_t} + \xi(Y_t) dN(\lambda)_t \end{cases}, t \in [0, T] \tag{27}$$

where $diag(S_t)$ is the diagonal matrix with S_t in the diagonal and 0 on the off-diagonal elements, $\eta, J \in \mathbb{R}^{N \times 1}$, $Z_t^{(1)} \in \mathbb{R}^{N \times 1}$ and $Z_t^{(2)} \in \mathbb{R}^{N \times N}$ are matrix Wiener processes, $K, Q, \xi \in \mathbb{R}^{N \times N}$ while $N(\lambda)_t$ is a non-compensated Poisson process with intensity $\lambda \in \mathbb{R}$. Furthermore Ω is a symmetric, positive definite and invertible matrix such that $\Omega\Omega' = \varphi QQ'$ with $\varphi \in \mathbb{R}$ and $\varphi > N - 1$. The Wiener processes determining shocks in prices and variance-covariance matrix $\Sigma_t = Y_t^{-1}$ are correlated according to

$$Z_t^{(1)} = Z_t^{(2)} \rho + \sqrt{1 - \rho'\rho} Z_t^{(3)}, \tag{28}$$

where $\rho \in \mathbb{R}^{N \times 1}$, $Z_t^{(3)} \in \mathbb{R}^{N \times 1}$is a Wiener process and the elements of $Z_t^{(3)}$ and $Z_t^{(2)}$ are all independent among them.

The approximated HJB for the investment problem is:

$$\max_{\phi}\Bigg(W_t+[\phi_t'(\eta-r\mathbf{1})+r]X_tW_X+Tr([\Omega\Omega'+KY_t+Y_tK']\nabla W)+\frac{1}{2}X_t^2\phi'Y_t^{-1}\phi_t$$
$$W_{XX}+(2\phi_t'\nabla Q'\rho W_X)\,X+\frac{1}{2}Tr\,(4Y_t\nabla Q'Q\nabla)\,W+\lambda\mathbb{E}[W(X_t+\phi'JX_t,Y_t+$$
$$\xi)-W(X_t,Y_t)]\Bigg), \tag{29}$$

where $\nabla := \left(\frac{\partial}{\partial Y_{i,j}}\right)_{1\leq i,j\leq N}$.
The value function associated with Equation (29) is given by:

$$W(t,Y_t,X_t)=\exp\{Tr(F_tY_t)+G_t\}\frac{X_t^{1-\gamma}}{1-\gamma}, \tag{30}$$

where Tr represents the trace of a square matrix. It's important to note that to recover a viable HJB equation in this realistic case the authors follow an approximated approach. The first approximation consists in assuming that the jump matrix in the co-precision ξ is constant, not depending on Y_t. The second approximation consists of the linearization of the jump term appearing in the HJB equation, using second-order Taylor expansion:

$$(1+\phi_t'J)^{1-\gamma}=1+(1-\gamma)\phi_t'J+o((\phi_t'J)^2), \tag{31}$$

and the approximation consists in not consider the $o((\phi_t'J)^2)$ term in the HJB equation.
The solution to the optimal allocation is dynamic consistently with the well-known Markowitz economic intuition, in that the optimal weight is proportional to the instantaneous co-precision:

$$\phi^*=Y_t\left[\frac{(\eta-r\mathbf{1})+2F_tQ'\rho+\lambda\mathbb{E}\left[e^{Tr(F_t\xi)}J\right]}{\gamma}\right]=:Y_tB_t, \tag{32}$$

where F_t solves

$$\dot{F}_t+(1-\gamma)(\eta)-r\mathbf{1}B_t'+F_tK+K'F_t-\frac{\gamma(1-\gamma)}{2}B_tB_t'+2(1-\gamma)F_tQ'\rho B_t' \tag{33}$$
$$+2F_tQ'QF_t+\lambda(1-\gamma)\mathbb{E}\left[e^{Tr(F_t\xi)}J\right]B_t'=0,F_T=0.$$

The optimal portfolio weights consist of three terms: the first one is the *myopic demand* component, which is now proportional to inverse volatility and so takes the typical form of standard mean-variance allocation; the second one is the *intertemporal hedging demand* term, which depends on the correlation coefficients and co-precision. Finally, the third term assumes the meaning of an illiquidity term.

Differently, Zhou et al. (2019) solves the dynamic portfolio allocation considering an AR(1)-GARCH(1,1)-ARJI model to describe the asset returns which enables capturing the dynamics process of both volatility and jump intensity. The authors find that both initial jump intensity and jump persistence are important for the investor's optimal portfolio decision.

More recently Jin et al. (2021) formulates a portfolio choice problem in a multi-asset market characterized by ambiguous jumps and constant volatility. In the paper, therefore, a portfolio choice problem with ambiguity and ambiguity aversion in a continuous-time incomplete financial market is considered. Recall that ambiguity is uncertainty that cannot be measured by a single probability measure. The authors investigate the impact of tail risk on portfolio selection. The optimal portfolio is solved in closed form through a decomposition approach, with two different jump size distributions. They also show that underestimating tail risk might result in a sizeable wealth loss in the presence of jump ambiguity. Furthermore, they confirm that ambiguity-averse investors reduce more of their jump exposure if the jump distribution exhibits a fatter left tail.

So far we have assumed an incomplete market setting. To complete the market, we can add some financial securities susceptible to the whole range of risk components. The natural choice is to include derivative contracts on the underlying portfolio equity. Among the numerous references we first focus on Liu and Pan (2003) who studied the impact of options on wealth allocation in a stochastic volatility framework considering only jumps in price. In particular, they consider the following dynamic for the risky asset $S = \{S_t\}_{t\in[0,T]}$:

$$\begin{cases} dS_t = (r + \eta V_t + J(\lambda - \lambda^{\mathbb{Q}})V_t)S_t dt + \sqrt{V_t} S_t dZ_t^{(1)} + JS_{t-}(dN_t - \lambda V_t dt) \\ dV_t = k(\theta - V_t)dt + \sigma\sqrt{V_t}\left(\rho dZ_t^{(1)} + \sqrt{1-\rho^2}dZ_t^{(2)}\right) \end{cases}, t \in [0, T] \tag{34}$$

where $Z_t^{(1)}$ and $Z_t^{(2)}$ are standard Brownian motions, and N is a pure-jump process. All the random shocks are assumed to be independent. The instantaneous variance process $V = \{V_t\}_{t\in[0,T]}$ is a stochastic process with long-run mean $\theta > 0$, mean-reversion rate $k > 0$, and vol-of-vol coefficient $\sigma \geq 0$. In this setting, the agent is interested in investing not only in stock and riskless bonds but also in derivatives. More specifically the derivatives involved are those providing different exposures to the three fundamental risk factors in the economy. The market can be completed once we introduce enough *non-redundant* derivatives $O_t^i = g^i(S_t, V_t)$ for $i = 1 \cdots N$ and the price dynamics for the i-th derivative security is:

$$\begin{aligned} dO_t^{(i)} =& rO_t^i dt + (g_s^{(i)} S_t + \sigma\rho g_v^{(i)})(\eta V_t dt + \sqrt{V_t} dB_t) + \sigma\sqrt{1-\rho^2} g_v^{(i)}(\epsilon V_t dt + \sqrt{V_t} dZ_t^{(1)} \\ &+ \Delta g^{(i)}(\lambda - \lambda^{\mathbb{Q}})V_t dt + dN_t - \lambda V_t dt, \end{aligned} \tag{35}$$

where $g_s^{(i)}$ and $g_v^{(i)}$ measure the sensitivity of the $i-$th price to infinitesimal changes in the stock price and volatility, respectively, and where $\Delta_g^{(i)}$ measures the change in the derivative price for each jump in the underlying stock price. To solve the investment problem, the authors define the indirect utility function

$$W(t, X_t, V_t) = \frac{X_t^{1-\gamma}}{1-\gamma} \exp\{\gamma F(T-t)V_t + \gamma G(T-t)\}. \tag{36}$$

which satisfies the following HJB equation,

$$\begin{aligned} \max_{\phi_t, \nu_t} \Bigg(& W_t + X_t W_X \left(r_t + \theta_t^B \eta V_t + \theta_t^Z \epsilon V_t - \theta_t^N J \lambda^Q V_t\right) + \frac{1}{2} X_t^2 W_{XX} V_t \left((\theta_t^B)^2 + (\theta_t^Z)^2\right) \\ & + \lambda V_t \Delta W + k(\theta - V_t)W_V + \frac{1}{2}\sigma V_t W_{VV} + \sigma V_t X_t W_{WV} \left(\rho\theta_t^B + \sqrt{1-\rho.^2}\theta_t^Z\right) \Bigg) \\ & = 0, \end{aligned} \tag{37}$$

where $\Delta W = W(t, X_t(1 + \theta^N J), V_t) - W(t, X_t, V_t)$ denotes the jump in the indirect utility function W for given jumps in the stock price, and where W_t, W_X and W_V denotes the derivatives of $W(t, X, V)$ with respect to t, X and V, similar notations for higher derivatives. The authors solve the optimization problem in closed form and the optimal portfolio weights on the risk factors B (bond), Z(stock), and N (derivatives) are given by:

$$\begin{aligned} \theta_t^{*B} &= \frac{\eta}{\gamma} + \sigma\rho F(\tau), \\ \theta^{*Z} &= \frac{\epsilon}{\gamma} + \sigma\sqrt{1-\rho^2}F(\tau,) \\ \theta^{*N} &= \frac{1}{J}\left(\left(\frac{\lambda}{\lambda^Q}\right)^{1/\gamma} - 1\right), \end{aligned} \tag{38}$$

where η and ϵ are the risk premia and

$$F(\tau) = \frac{exp(k_2\tau) - 1}{2k_2 + (k_1 + k_2)(exp(k_2\tau) - 1)}\delta, \tag{39}$$

with

$$\begin{aligned} \delta &= \frac{1-\gamma}{\gamma}(\eta^2 + \epsilon^2) + 2\lambda^{\mathbb{Q}}\left[\left(\frac{\lambda}{\lambda^{\mathbb{Q}}}\right) + \frac{1}{\gamma}\left(1 - \frac{\lambda}{\lambda^{\mathbb{Q}}}\right) - 1\right], \\ k_1 &= k - \frac{1-\gamma}{\gamma}\left(\eta\rho + \epsilon\sqrt{1-\rho^2}\right)\sigma, \\ k_2 &= \sqrt{k_1^2 - \delta\sigma^2}. \end{aligned}$$

The optimal exposure to the three risk factors, according to Liu et al. (2003) for the incomplete market setting, does not depend on instantaneous volatility. The authors showed as derivatives extend the risk and return tradeoffs associated with stochastic volatility and price jumps. In particular, they illustrated two significant examples. In the first example, they focused on the role of derivatives as a vehicle for volatility risk and as result, the optimal portfolio weight in derivative security depends explicitly on the sensitivity of the derivative to volatility. The second example is the role of derivatives as a vehicle to disentangle jump risk from diffusive risk.

Later Branger et al. (2007) extended the Liu and Pan (2003) framework by also considering discontinuities in the stochastic process that governs volatility. They showed that the demand for jump risk includes a hedging component which is not present in the models without volatility jump, this is the main difference with the previous setting. More precisely, we have

$$\theta^{*N} = \frac{1}{J}\left[\left(\frac{\lambda}{\lambda^{\mathbb{Q}}}\right)^{1/\gamma} - 1\right] + \frac{1}{J}\left(\frac{\lambda}{\lambda^{\mathbb{Q}}}\right)\left[e^{F(\tau)\xi} - 1\right], \tag{40}$$

with ξ volatility jump size assumed to be constant, and consequently this impacts also on $F(\tau)$, equation (39), while the other exposures $(\theta^{*B}, \theta^{*Z})$ are the same as Equation (38) of Liu and Pan (2003).

Moreover, the authors showed how the volatility jump magnitude has a significant impact on the optimal portfolio. They analyzed the distribution of terminal wealth for an investor who uses the wrong model by ignoring volatility jumps or including wrong estimates of such jumps.

In this context, we also mention Escobar et al (2015), who determine the optimal portfolio for an ambiguity-averse investor when stock price follows a stochastic volatility jump-diffusion process (considering only a jump in the stock dynamics), and when the investor can have different levels of uncertainty about diffusion parts of the stock and its volatility. The authors illustrate that the optimal exposures to stock and volatility risks are significantly affected by ambiguity aversion to the corresponding risk factor only. Moreover, they also show that volatility ambiguity has a smaller impact in incomplete markets. As an extension, Cheng and Escobar-Anel (2021) consider an optimal allocation problem with both risk and ambiguity aversion under a 4/2 model. The numerical analysis finds that *wealth-equivalent losses* (WELs) from ignoring uncertainty or market completeness are moderate, while the WELs for investors who follow different models such as Heston or Merton (geometric Brownian motion) is quite substantial.

Unlike previous works in Ilhan and Sicar (2005) the investors maximize *expected exponential utility* function of terminal wealth and restrict a static position in derivative securities. The main result is that in a general incomplete arbitrage-free market there exists a unique optimal strategy for the investor.

Haugh and Lo (2001) construct a buy-hold portfolio of stocks, bonds, and options that involves no trading once set at the beginning of the investment horizon, and solve this problem for several combinations of preferences as CRRA and CARA (*Constant Absolute Risk Aversion*) preferences and different return dynamics (*Geometric Brownian motion*, the *Ornstein Uhlenbeck process* and a *Bivariate Linear Diffusion process*). The authors show that under certain conditions a portfolio consisting of just a few options is an excellent substitute for more complex dynamic policies.

3.2 Non-expected utility: Epstein–Zin preferences

As an alternative to power utilities, a strand of extant literature has focused on *Epstein-Zin preferences*, since they include the effect of both risk aversion, and separate EIS from the coefficient of relative risk aversion. The power utility functions restrict risk aversion to be the reciprocal of the elasticity of intertemporal substitution and this does not reflect the empirical evidence that has shown how these parameters have very different effects on optimal consumption and portfolio choice, as highlighted in Chacko and Viceira (2005). The latter examines the optimal consumption and portfolio-choice problem of long-horizon investors in an incomplete market setting, by first introducing *precision* process, intended as the inverse of volatility, to obtain dynamic optimal portfolio rules. The authors consider recursive utility over consumption and derive an analytic expression for the optimal consumption and portfolio policies. The latter are exact when an investor has unit elasticity of intertemporal substitution of consumption, and approximate otherwise.

To simplify the analysis, the market is assumed to be made of only two assets, the first one is the riskless asset with dynamics:

$$\frac{dB_t}{B_t} = rdt, \tag{41}$$

where r is the risk-free rate. The second one is stock, with the following dynamics :

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{y_t^{-1}} S_t dZ_t^{(1)} \\ dy_t = k(\theta - y_t)dt + \sigma\sqrt{y_t} dZ_t^{(2)} \end{cases}, t \in [0, T] \tag{42}$$

where y_t is the instantaneous precision of the risky asset return that follows a mean-reverting process, with $\frac{1}{v_t} = y_t$, being v_t the variance process, and $k, \theta, \sigma > 0$. Moreover in such a setting shocks to precision are correlated with the instantaneous returns on the risky asset, with $\langle dZ_t^{(1)} dZ_t^{(2)} \rangle = \rho dt$. The HJB equation for this problem is:

$$\sup_{\phi_t, C} \left(f(C_s, J_s) + \left[\phi_t(\mu - r)X_t + rX_t - C_t\right] W_X + \frac{1}{2}\phi_t^2 X_t^2 W_{XX} \frac{1}{y_t} + k(\theta - y_t)W_y + \right.$$
$$\left. \frac{1}{2}\sigma^2 W_{yy} y_t + \rho\sigma\phi_t X_t W_{Xy} \right) = 0, \tag{43}$$

where $f(C,J) = \beta(1-\gamma)W\left[\log(C) - \frac{1}{1-\gamma}\log((1-\gamma)W)\right]$ is the aggregator for $\psi = 1$ and subscripts on W denote partial derivatives. When $\psi = 1$, there is an exact analytical solution to the optimization problem with value function given by:

$$W(X_t, y_t) = \frac{X_t^{1-\gamma}}{1-\gamma}\exp\{Fy_t + G\}, \tag{44}$$

where F_t and G_t are deterministic functions to be determined. This value function implies the following optimal consumption and portfolio rules:

$$\begin{cases} \frac{C_t}{X_t} = \beta \\ \phi_t^* = \frac{1}{\gamma}(\mu - r)y_t + \left(1 - \frac{1}{\gamma}\right)(-\rho)\sigma F y_t \end{cases}, \tag{45}$$

with $\beta > 0$ rate of time preferences, $\gamma > 0$ is the coefficient of relative risk aversion, and F_t, G_t are given by the solution of the following system:

$$aF^2 + bF + c = 0, \tag{46}$$

$$(1-\gamma)(\beta\log\beta + r - \beta) - \beta G + k\theta F = 0 \tag{47}$$

where

$$a = \frac{\sigma^2}{2\gamma(1-\gamma)}[\gamma(1-\rho^2) + \rho^2], b = \frac{\rho\sigma(\mu - r)}{\gamma}, c = \frac{(\mu-r)^2}{2\gamma}. \tag{48}$$

When $\psi \neq 1$ the HJB is still given by (43) but the first-order condition for consumption is different due to the different form of the aggregator, as in equation (12). So, in this case, guessing that

$$W(X_t, y_t) = I(y_t)\frac{X_t^{1-\gamma}}{1-\gamma} \tag{49}$$

with the transformation $I = H^{\frac{-1-\gamma}{1-\psi}}$ and replace into HJB equation, also replacing the expression for ϕ_t, the authors obtain a *non-homogeneous* ordinary differential equation:

$$\begin{aligned} &-\beta^\psi H^{-1} + \phi\beta + \frac{(1-\psi)(\mu-r)^2}{2\gamma}y_t - \frac{\rho\sigma(\mu-r)(1-\gamma)}{\gamma}\frac{H_y}{H}y_t + r(1-\psi) + \\ &\frac{\rho^2\sigma^2(1-\gamma)^2}{2\gamma(1-\psi)}\left(\frac{H_y}{H}\right)^2 y_t - \frac{H_y}{H}k(\theta - y_t) + \frac{\sigma^2}{2}\left(\frac{1-\gamma}{1-\psi} + 1\right)\left(\frac{H_y}{H}\right)^2 y_t \frac{\sigma^2}{2}\frac{H_{yy}}{H}y_t = 0, \end{aligned} \tag{50}$$

the solution is obtained by approximating the term

$$\beta^\psi H^{-1} = \exp\{c_t - x_t\}, \tag{51}$$

where $c_t - x_t = \log(C_t/X_t)$ and using a first-order Taylor expansion of $\exp\{c_t - x_t\}$:

$$\beta^\psi H^{-1} \approx h_0 + h_1(c_t - x_t),$$

where $h_1 = \exp\overline{c - x}$ and $h_0 = h_1(1 - \log h_1)$. Substituting Equation (51) in the first term of Equation (50), the resulting ODE has a solution of the form $H = \exp\{F_1 y_t + G_1\}$. This solution implies the following value function

$$W(X_t, y_t) = \frac{X_t^{1-\gamma}}{1-\gamma}\exp\left\{-\left(\frac{1-\gamma}{1-\psi}\right)(F_1 y_t + G_1)\right\}. \tag{52}$$

In this latter case the optimal consumption is equal to $\frac{C_t}{X_t} = \beta^{\psi} e^{-F_1 y_t - G_1}$, F_1 and G_1 are given by solution to a system similar to (46), see Appendix A. of Chacko and Viceira (2005) for more details.

The optimal portfolio rule has two components, namely a *myopic portfolio demand*, and a Merton's *intertemporal hedging demand*. Both components are linear functions of precision. The optimal consumption wealth ratio is invariant to changes in volatility if $\psi = 1$, while it is an affine function of instantaneous precision when $\psi \neq 1$. In a similar framework, Faria and Correia-da-Silva (2016) extended the model of Chacko and Viceira (2005) for optimal dynamic portfolio choice, introducing ambiguity in stochastic investment opportunity set, showing a small impact of the elasticity of intertemporal substitution of consumption (EIS) on optimal allocation. Since standard verification results are not applicable (Duffie and Epstein (1992)), due to the non-Lipschitzianity of Epstein-Zin preferences, Kraft et al. (2013) provided a suitable verification theorem for the associated HJB equation. This paper contributes to providing new explicit solutions to the HJB equation with recursive utility for a non-unit EIS. Those results represent the first explicit benchmark for the *Cambell-Shiller approximation*, used by Chacko and Viceira (2005) in their approximation. Kraft et al. (2016) extended their previous work and provided the existence and uniqueness of solutions of HJB equation by exploiting fixed point arguments, and developed a fast and accurate numerical method for computing both indirect utility and optimal strategies.

Xing (2017) studied an investment problem via backward stochastic differential equations considering a multi-asset model in which the assets follow a Geometric Brownian motion and volatility is constant, focusing on the empirically relevant specification where both risk aversion and EIS are greater than one. The utility specification makes the optimization problem very difficult to solve since the Epstein-Zin aggregator is not Lipschitzian.

In a complete market setting, we refer to Hsuku (2007), where a recursive utility function defined on intermediate consumption (rather than terminal wealth) is maximized, to reflect the realistic behavior of an investor who saves money for the future. This setting is based on a general assumption according to which expected stock returns are affine functions of volatility. The economy is formed by a riskless bond

$$\frac{dB_t}{B_t} = r dt, \tag{53}$$

where r is the risk-free rate and a risky stock with the following dynamic:

$$\begin{cases} \frac{dS_t}{S_t} = \mu dt + \sqrt{V_t} dZ_t^{(1)} \\ dV_t = k(\theta - V_t)dt + \sigma\sqrt{V_t}\rho dZ_t^{(1)} + \sqrt{1-\rho^2} dZ_t^{(2)} \end{cases}, t \in [0, T] \tag{54}$$

where $Z_t^{(1)}$ and $Z_t^{(2)}$ are standard Brownian motions. In such a market, the authors refers to Liu and Pan (2003), so that the non-redundant derivative $O_t = g(S_t, V_t)$ at the time $t \in [0, T]$ has the following dynamics:

$$dO_t = \left[(\mu - r)(g_s S_t + \rho\sigma g_v) + \eta\sigma\sqrt{1-\rho^2} g_v + r O_t\right] dt + (g_s S_t + \rho\sigma g_v)\sqrt{V_t} Z_t^{(1)} + (\sigma\sqrt{1-\rho^2} g_v)\sqrt{V_t} dZ_t^{(2)}, t \in [0, T] \tag{55}$$

where η determines the stochastic volatility risk premium, g_s and g_v are measures of derivative price sensitivities to small changes in the underlying stock price and volatility, respectively. When $\psi \to 1$

the optimal portfolio weights in stock and derivatives, and the optimal consumption wealth ratio is:

$$\frac{C_t}{X_t} = \beta, \tag{56}$$

$$\phi_t^* = \frac{1}{\gamma}\frac{\mu - r}{V_t} + \frac{1}{\gamma}\left(Q_1 + Q_2\right)\frac{1}{V_t})\rho\sigma - \frac{1}{\gamma}\frac{\eta}{V_t}\frac{(g_s S_t + \rho\sigma g_v)}{\sigma\sqrt{1-\rho^2}g_v} - \frac{1}{\gamma}\left(Q_1 + Q_2\frac{1}{V_t}\right) \tag{57}$$
$$\frac{(g_s S_t + \rho\sigma g_v)}{g_v},$$

$$\nu^* = \frac{1}{\gamma}\frac{\lambda}{V_t}\frac{1}{\sigma\sqrt{1-\rho^2(g_v/O_t)}} + \frac{1}{\gamma}\left(Q_1 + Q_2\frac{1}{V_t}\right)O_t/g_v, \tag{58}$$

with Q_1 and Q_2 that solve the following equations:

$$\left(\frac{1}{2}\frac{1}{1-\gamma}\sigma^2 + \frac{1}{2}\frac{1}{\gamma}\sigma^2\right)Q_2^2 + \left[\frac{1}{\gamma}\sigma(\mu - r)\rho\lambda\sqrt{1-\rho^2} + \frac{1}{1-\gamma}k\theta - \frac{1}{2}\frac{1}{1-\gamma}\sigma^2\right]Q_2 + \tag{59}$$
$$\frac{1}{2}\frac{1}{\gamma}[(\mu - r)^2 + \lambda^2] = 0, \left(\frac{1}{2}\frac{1}{1-\gamma}\sigma^2 + \frac{1}{2}\frac{1}{\gamma}\sigma^2\right)Q_1^2 - \left(\frac{1}{1-\gamma}\beta + \frac{1}{1-\gamma}k\right)Q_1 -$$
$$\frac{1}{1-\gamma}\beta\frac{1}{\theta}Q_2 = 0.$$

For the general case $\psi \neq 1$ there is no exact solution. To find an approximate solution the authors apply the approximation proposed in Chacko and Viceira (2005). Equation (56) demonstrates the invariance of the optimal log consumption–wealth ratio to changes in volatility when $\psi = 1$, as already seen in Chacko and Viceira (2005) . The results obtained in Hsuku (2007) further show that optimal consumption–wealth ratio is a function of stochastic volatility when $\psi \neq 1$: in particular it is an increasing function for investors whose EIS is smaller than one, while it is a decreasing function for investors whose EIS is larger than one. Finally, the analyses confirm the conclusions of Liu and Pan (2003) in the case of the power utility, regarding the role of derivatives: derivative securities are a significant tool for expanding investors' dimension of risk-and-return tradeoffs, being a vehicle for the additional risk factor of stochastic volatility in the stock market.

4 Conclusions

In this paper, a review of the literature on dynamic allocation has been proposed. In particular, some of the most popular models for the investor's choice strategy were analyzed, considering different preferences for the investor and market settings. It is observed how the use of affine models to model stochastic volatility leads to optimal weights that do not depend on the instantaneous volatility nor on the long-run mean level contradicting the foundations of the Markowitz portfolio theory, which theorize and demonstrates a relationship of inverse proportionality between the invested wealth and the synthetic indicators of variability.

Further contributions could be extended along two different lines: at first, the problem of optimal allocation in a complete market could be faced by considering the co-precision (the inverse of volatility) in a setting where there are discontinuities in both price and volatility to understand the impact of jumps and derivatives on the optimal portfolio in presence of precision when the investor has a power utility; at second the optimal allocation problem can be solved considering non-separable preferences for the investor (SDU utility function) and discontinuities in both price and volatility, analyzing both cases of complete and incomplete market.

References

Ait-Sahalia Y., Cacho-Diaz J., and Hurd T.R. (2009), Portfolio choice with jumps: a closed-form solution, *The Annals of Applied Probability*, 10(2), 556-584.

Branger N., Schlag C. and Schneider E. (2007), Optimal Portfolios when volatility can jump, *Journal of Banking and Finance*, 32(6), 1087–1097.

Buraschi A., Porchia P. and Trojani F. (2010), Correlation risk and optimal portfolio choice, *The Journal of Finance*, 65(1), 393-420.

Carr P., Jin X. and Madan D.B. (2001), Optimal investment in derivative securities, *Finance and Stochastics*, 4, 33-59.

Chacko G. and Viceira L.M. (2005), Dynamic Consumption and Portfolio Choice with Stochastic Volatility in Incomplete Market, *Review of Financial Studies*, 18(4),1369–1402.

Cheng Y. and Escobar-Anel M. (2021), Robust portfolio choice under the $4/2$ stochastic volatility model, *IMA Journal of Management Mathematics*, 10.

Cox J.C. and Huang C.F. (1989), Optimal consumption and portfolio policies when asset prices follow a diffusion process, *Journal of Economic Theory*, 49, 33-83.

Das S. R. and Uppal R. (2004), Systemic Risk and International Portfolio Choice, *Journal of Finance*, 59 (6), 2809–2834.

Duffie D. and Epstein L.G. (1992), Stochastic differential utility, *Econometrica*, 60(2), 353-294.

Duffie D., Pan J. and Singleton K. (2000), Transform analysis and asset pricing for affine jump-diffusions, *Econometrica*, 16(68), 1343-1376.

Epstein L.G. and Zin S.E. (1989), Substitution, risk aversion, and the temporal behavior of consumption and asset returns: a theoretical framework, *Econometrica*, 57(4), 937-969.

Escobar M., Ferrando S. and Rubtsov A. (2015), Robust portfolio choice with derivative trading under stochastic volatility, *Journal of Banking &Finance*, 61(2015), 142-157.

Faria G. and Correia-da-Silva J. (2016), Is stochastic volatility relevant for dynamic portfolio choice under ambiguity?, *The European Journal of Finance*, 22(7), 601-626.

Fisher M. and Gilles C. (1999), Consumption and Asset Prices with Homothetic Recursive Preferences, *Federal Reserve Bank of Atlanta*, 22(7), Working Paper 99-17.

Fleming W.H. and Hernandez-Hernandez D. (2003), An optimal consumption model with stochastic volatility, *Finance and Stochastics*, 7, 245-262.

Hansen L. P. and Singleton K. J. (1982), Generalized instrumental variables estimation of nonlinear rational expectations models, *Econometrica*, 50(5), 1269-1286.

Haugh M. and Lo A. (2001), Asset allocation and derivatives, *Quantitative Finance*, 1, 45-72.

Hsuku (2007), Dynamic consumption and asset allocation with derivative securities, *Quantitative Finance*, 7(2), 137-149.

Ilhan A. and Sicar R. (2005), Optimal investment with derivative securities, *Finance and Stochastics*, 9, 585-595.

Jin X. and Zhang Allen X. (2012), Decomposition of optimal portfolio weight in a jump-diffusion model and its applications, *The Review of Financial Studies*, 25(9), 2877-2919.

Jin X., Luo D. and Zeng X. (2021), Tail risk and robust portfolio decisions, *Management Science*, 67(5), 3254-3275.

Karatzas I., Lehoczky J.P. and Shreve S. (1987), Optimal portfolio and consumption decisions for a small investor on a finite horizon, *SIAM J. Control Optimization*, 25, 1557-1586.

Kraft (2005), Optimal portfolios and Heston's stochastic volatility model: an explicit solution for power utility, *Quantitative Finance*, 5(3), 303-313.

Kraft H., Seifried F.T. and Steffensen M. (2013), Consumption-portfolio optimization with recursive utility in incomplete markets, *Finance and Stochastics*, 17, 161-196.

Kraft H., Seifried F.T. and Steffensen M. (2016), Optimal consumption and investment with Epstein-Zin recursive utility, *Finance and Stochastics*, 21, 187-226.

Liu J. and Pan J. (2003), Dynamic derivative strategies, *Journal of Financial Economics*, 69, 401-430.

Liu J., Longstaff A. and Pan J. (2003), Dynamic Asset Allocation with Event Risk, *The Journal of Financial Economics*, 58(1), 231-259.

Liu J. (2007), Portfolio selection in stochastic environments, *The Review of Financial Studies*, 20(1), 1-39.

Markowitz H. (1952), Portfolio Selection, *Journal of Finance*, 7, 77-91.

Merton R.C. (1971), Optimum consumption and portfolio rules in a continuous-time model, *Journal of Economic Theory*, 3(4), 373-413.

Oliva I. and Renò R. (2018), Optimal portfolio allocation with volatility and co-jump risk that Markowitz would like, *Journal of Economics Dynamics and Control*, 94, 242-256.

Pharm H. (2022), Smooth solutions to optimal investment models with stochastic volatilities and portfolio constraints, *Applied Mathematics and Optimization*, 46, 55-78.

Pharm H. (2007), Optimization methods in portfolio management and option hedging,*3rd cycle. Hanoi(Vietnam)*, 27, cel-00392212f.

Plisca R.S (1986), A stochastic calculus model of continuous trading: Optimal portfolios,*Math. Operations Res*, 11, 371-382.

Schroder M. and Skiadas C. (1999), Optimal Consumption and Portfolio Selection with Stochastic Differential Utility,*Journal of Economic Theory*, 89, 68-126.

Schroder M. and Skiadas C. (2003), Optimal Lifetime Consumption-Portfolio Strategies under Trading Constraints and generalized Recursive Preferences, *Stochastic Processes and their Applications*, 108, 155-202.

Von Neuman J. and Morgenstern O. (1947), Theory of games and economic behavior, *Princeton University Press*.

Xing H. (2017), Consumption–investment optimization with Epstein–Zin utility in incomplete markets,*Finance Stochastics*, 21, 227–262.

Zhou C., Wu C. and Wang Y. (2019), Dynamic portfolio allocation with time-varying jump risk, *Journal of Empirical Finance*, 50, 113-124.