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Denis Veliu*

ENHANCING RISK PARITY MODELS: A TWO-STAGE APPROACH USING MEAN-VARIANCE AND EXPECTED SHORTFALL FOR OPTIMAL ASSET SELECTION

Abstract

In a progressively complex and volatile financial scenery, the need for robust investment strategies has never been greater. Portfolio optimization seeks to balance risk and return for superior performance. Traditional approaches, such as the Mean-Variance model by Markowitz [1952], laid the groundwork for diversification and risk management. Risk Parity models offer a compelling alternative by equalizing risk contributions across assets, but often lack an optimal mechanism for asset selection, leading to suboptimal results, especially with large and diverse asset universes. This paper proposes a novel two-stage approach. First, we use the Mean-Variance model and Expected Shortfall (ES) to select a refined subset of assets, minimizing risk without imposing return constraints. Second, we apply Risk Parity techniques—standard deviation for the Mean-Variance subset, ES for the ES-selected subset—to balance risk contributions. Tested on both Nikkei 225 equities and a mixed portfolio of stocks, bonds, and cryptocurrencies, our method enhances resilience while preserving returns.

Keywords: Risk Parity, Expected Shortfall, Asset Selection, Diversification, Portfolio Optimization

*Universiteti Metropolitan Tirana, Albania

1 Introduction

There is a paradox of choice and complexity in the modern investment environment. Although the variety of asset classes available to modern portfolios is unprecedented, ranging from traditional equities and fixed income to cryptocurrencies and derivatives, this expansion has brought about new aspects of risk and correlation that put traditional optimization frameworks to the test [Markowitz, 1952, Platanakis and Urquhart, 2020]. The 2008 global financial crisis exposed the shortcomings of conventional portfolio design techniques, especially their susceptibility to non-normal return distributions and tail-risk occurrences [Artzner and Delbaen, 1999, Rockafellar and Uryasev, 2000].

Risk Parity techniques developed as a tempting alternative to mean-variance optimization, suggesting more balanced risk allocation across assets [Qian, 2005, Maillard et al., 2009]. These methods, however, frequently overlook a crucial first-order issue: deciding which assets should be included in the portfolio universe. In their examination of bitcoin portfolios, Veliu and Aranitasi [2024] showed that careless asset inclusion might result in unanticipated risk concentrations and less than ideal diversification. When working with diverse asset universes, whose correlation structures might be unstable or poorly understood, this oversight becomes especially problematic [Boiko et al., 2021].

Our research addresses this gap by introducing a novel two-stage optimization framework that combines rigorous asset selection with dynamic risk allocation. Building upon the theoretical foundations laid by Tasche [2000] and Stefanovits [2010], we first employ conditional value-at-risk (CVaR) metrics to identify assets with favorable risk-return characteristics, then apply modified Risk Parity techniques to the refined universe. This dual-phase approach operationalizes several key insights from recent literature:

1. The importance of tail-risk management in portfolio construction [Andersson et al., 2000]
2. The value of selective diversification over naive inclusion [Ma et al., 2020]
3. The need for adaptive methodologies in evolving markets [Bjerring et al., 2016]

Empirical testing across both traditional equity markets (Nikkei 225) and mixed-asset portfolios demonstrates the framework's robustness. Our results align with and extend the findings of Bacon [2008] on performance measurement and Sharpe [1966] on risk-adjusted returns, while addressing the implementation challenges identified by Colucci [2013].

This study makes three primary contributions to the literature:

1. An operationalizable solution to the asset selection problem in Risk Parity strategies
2. Empirical evidence of improved risk-adjusted performance across market regimes
3. A replicable framework for incorporating alternative assets into traditional portfolios

As financial markets continue to evolve with technological innovation and regulatory change, our findings offer both theoretical insights and practical tools for navigating this complex landscape. The methodology's success in managing the unique characteristics of cryptocurrencies [Segendorf, 2014] while maintaining robustness in traditional asset classes suggests its potential as a unifying approach for 21st-century portfolio construction.

2 Methodology

Consider a portfolio composed by n assets and weights $w = (w_1, w_2, \ldots, w_n)$, the standard deviation is given by

$$\sigma_P(w) = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}} = \sqrt{w'\Omega w}\ , \tag{1}$$

where Ω is the variance-covariance matrix of the returns. If the vector of returns is the mean of each asset $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ then the portfolio return is

$$\mu_P = \mu' w = \sum_{i=1}^{n} w_i \mu_i\ . \tag{2}$$

The expected shortfall or Conditional Value at Risk (CVaR) can be calculated as follows:

$$ES_\alpha(w) = \mathbb{E}[-\mu_P \mid \mu_P \leq -VaR_\alpha(w)]\ , \tag{3}$$

or, in continuous form, by the definition of ES_α (uryasev, 2000), we have

$$ES_\alpha(w) = \frac{1}{\alpha}\int_0^\alpha VaR_v(w)\, dv\ . \tag{4}$$

	Markowitz model	**ES (10%)**
Objective	$\min \sigma_p^2$	$\min \mathrm{ES}$
Expected Return	$\mu_P = \sum_{i=1}^{n} w_i \mu_i$ (not applied)	$\mu_P = \sum_{i=1}^{n} w_i \mu_i$
Weights Sum	$\sum_{i=1}^{n} w_i = 1$	$\sum_{i=1}^{n} w_i = 1$
Constraints	$w \geq 0$ (short selling n.a.)	$w \geq 0$

Table 1. Comparison of portfolio optimization models

In step one we apply only the Mean Variance and Expected Shortfall models to find the optimal weights w. At this juncture, performance is at its lowest; this step is only used to identify the asset group inside a vast category.

Let us call the elements of the selected subset Λ and their corresponding vector of weights $y = (y_1, y_2, \ldots, y_m)$, for which we will apply the Risk Parity models in the second phase.

The weights in the first phase are assigned as follows:

$$y_i^t = \begin{cases} w_i^t & \text{if } i \in \Lambda \text{ and } w_i^t > \varepsilon \\ 0 & \text{if } i \notin \Lambda \text{ or } w_i^t < \varepsilon \end{cases}\ ,$$

where $\varepsilon = 10^{-6}$ is the threshold to decide whether to include an asset or not. This means that if the weight selected at the first stage is less than ε, then it is excluded from the subset for the second phase. Typically, the subset size m satisfies $m \leq n$ compared to the original set of n assets.

In the second stage, we implement Risk Parity utilizing the standard deviation for the selected group based on the Mean-Variance model, and apply Risk Parity with Expected Shortfall for the group selected using Expected Shortfall.

The Equally Risk Contribution (Risk Parity condition) using the standard deviation as risk measure is given by:

$$TRC_i(y) = y_i \frac{\partial \sigma_P(y)}{\partial y_i} = y_i \frac{\partial \left(\sigma_i^2 + \sum_{j=1}^{n} y_j \sigma_{ij} \right)}{\sigma_P(y) \partial y_i} = y_i \frac{(\Omega y)_i}{\sqrt{y' \Omega y}} \ , \tag{5}$$

and the sum of total risk contributions satisfies:

$$\sum_{i=1}^{m} TRC_i(y) = \sum_{i=1}^{n} y_i \frac{(\Omega y)_i}{\sqrt{y' \Omega y}} = \sqrt{y' \Omega y} = \sigma_P(y) \ . \tag{6}$$

The Risk Parity model can then be formulated as the following optimization problem:

$$y^* = \arg\min_y \sum_{i=1}^{m} \sum_{j=1}^{m} \left(TRC_i(y) - TRC_j(y) \right)^2 \ , \tag{7}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} y_i = 1 \ , \tag{8}$$

$$y \geq 0 \ . \tag{9}$$

Since $TRC_i(y) = y_i \frac{\partial \sigma_P(y)}{\partial y_i}$ and $TRC_i(y) = \frac{\sigma_P(y)}{m}$, the problem equivalently is:

$$y^* = \arg\min_y \sum_{i=1}^{m} \sum_{j=1}^{m} \left(TRC_i(y) - TRC_j(y) \right)^2 \ , \tag{10}$$

subject to constraints (8) and (9).

The same approach applies starting from the Expected Shortfall $ES_\alpha(x)$, which is equivalent to $CVaR_\alpha(x)$, as Tasche [2000] and Stefanovits [2010] showed under the condition $\mathbb{E}[X^-] < \infty$.

The Total Risk Contribution for each asset i of the portfolio is:

$$TRC_i^{ES_\alpha}(y) = y_i \frac{\partial ES_\alpha(y)}{\partial y_i} \ , \tag{11}$$

and in case of continuous returns distribution:

$$TRC_i^{ES_\alpha}(y) = -y_i \mathbb{E} \left[\mu_i y_i \mid \mu_i y_i \leq -VaR_\alpha(y) \right] \ , \tag{12}$$

with

$$ES_\alpha(y) = \sum_{i=1}^{n} TRC_i^{ES_\alpha}(y) = -\sum_{i=1}^{n} y_i \mathbb{E} \left[\mu_i y_i \mid \mu_i y_i \leq -VaR_\alpha(y) \right] \ . \tag{13}$$

Additionally, Risk Parity with Expected Shortfall is used to determine which group has the lowest global riskiness after initial selection by Expected Shortfall.

For numerical approximation, we use the method of Stefanovits [2010] proposed in his master's thesis, where he applies a Gaussian kernel estimate to implement equally risk contribution in the case of standardized multivariate distribution. Using a parametric method, he applied Risk Parity to Expected Shortfall assuming normal or Student-t distributions. We also include the Uniform or naive portfolio to grasp the concept. To enhance portfolio return, we compute the compound return for the period.

For diversification measure, we use and graphically show the Herfindahl Index:

$$D = 1 - \sum_{i=1}^{n} w_i^2 \; . \tag{14}$$

3 Results of the Empirical Research

3.1 Case 1: The large set of stocks of Nikkei225

We selected an observation period of 756 weeks, or 174 months (14.5 years), from 1/1/2000 to 4/7/2014 to understand how these models perform during the global crisis of 2008. The data stream THOMSON REUTERS provides the data at weekly frequency, referring to the adjusted closing prices. Due to halted series or incomplete data, we do not list all titles but select 188 out of the possible 225. To examine the performance of Risk Parity strategies outside of the sample, groups with varying asset counts are chosen.

We calculate the Uniform, the Mean-Variance and Expected Shortfall at the global minimum, the Risk Parity with standard deviation, the Risk Parity with Expected Shortfall, and the Risk Parity with Expected Shortfall naïve (worst case scenario). In every situation, we use models without leverage or short selling. We also disregard weights less than 10^{-6}. The case study concerns the assets of Nikkei225. We consider 188 assets out of 225 due to missing data. We create a rolling time window with in-sample period $L = 4$ years and out-of-sample period $H = 4$ weeks.

Table 2. The performance of the portfolios with assets of NIKKEI225 on the first selection

	R.P. STD	M-V	R.P. ES.N	R.P. ES	$\text{ES}_{10\%}(x)$	Uniform
$\mu(\%)$	0.0405	0.0031	0.0446	0.0357	0.0272	0.0449
$\mu_{\text{ann}}(\%)$	2.1286	0.1628	2.3472	1.8741	1.4255	2.3621
$\mu^c(\%)$	-2.518	-13.66	-1.535	-4.234	-0.471	-4.359
$\sigma(\%)$	2.9439	2.3808	3.0211	2.8927	2.3160	3.1991
S_σ	0.1003	0.0095	0.1077	0.0898	0.0854	0.1024

Source: Computation in Matlab

The Uniform and Risk Parity with Expected Shortfall Naïve strategies that do not consider risk, or take the worst strategy, also lead in annualized performance (2.3621%), highlighting their strength over longer-term horizons. The compound returns $\mu^c(\%)$ are negative across all strategies, reflecting performance penalties when considering risks.

Mean-Variance shows the most significant underperformance ($\mu^c(\%) = -13.66\%$), possibly due to its higher reliance on variance minimization, which may lead to overly conservative allocations.

Standard deviation is highest for Uniform (3.1991%), suggesting that while it delivers higher returns, it also carries more risk. Risk Parity approaches (e.g., "RP STD") maintain lower volatility (2.9439%), aligning with their focus on risk balance. Risk-Return Balance given by Sharpe ratio is highest for Risk Parity with Expected Shortfall (0.1077), showing better risk-adjusted returns compared to other strategies.

The Uniform strategy, despite its high returns, has a relatively moderate Sharpe ratio, indicating room for improvement in risk management.

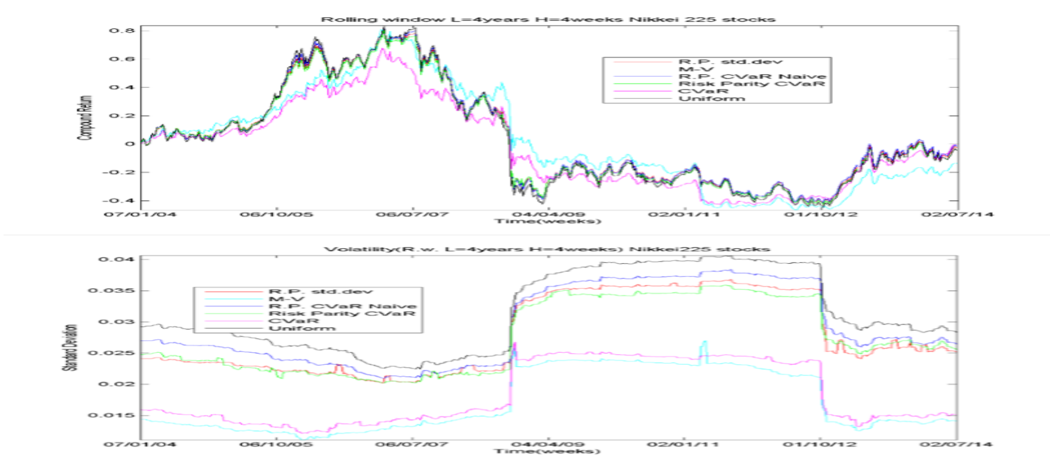

Figure 1. Performance out of sample for Nikkei225 (188 stocks) with the first selection (top chart) and volatility out of sample for Nikkei225 (188 stocks) with the first selection (bottom chart)

Prior to the crisis, as there are more assets, the Risk Parity models have a greater drawdown during the crisis, but still outperform CVaR. As before, the riskiness hierarchy remains the same.

It is interesting to see the number of assets each model focuses on:

The bottom graph in Fig. 1 illustrates the volatility of various strategies (measured by standard deviation) for the Nikkei 225 stocks over time.

- **Risk Parity (R.P.) Std. Dev:** Maintains relatively stable volatility over time, with slight increases during turbulent periods.
- **Mean-Variance (M-V):** Similar pattern to Risk Parity, showing moderate volatility.
- **Uniform:** Exhibits the highest volatility, particularly during the financial crisis (2008–2009).
- **R.P. CVaR Naive and Risk Parity CVaR:** Show consistently lower volatility, highlighting their focus on minimizing downside risk.
- **Expected Shortfall (CVaR):** Demonstrates the lowest and most stable volatility throughout.

A sharp spike in volatility is observed around the 2008 global financial crisis, across all strategies. "Uniform" experiences the highest volatility spike, indicating greater exposure to market risks. Post-crisis, volatilities normalize, though CVaR strategies recover faster. CVaR-based strategies (R.P. CVaR Naive, Risk Parity CVaR, CVaR) effectively minimize volatility, especially during market stress.

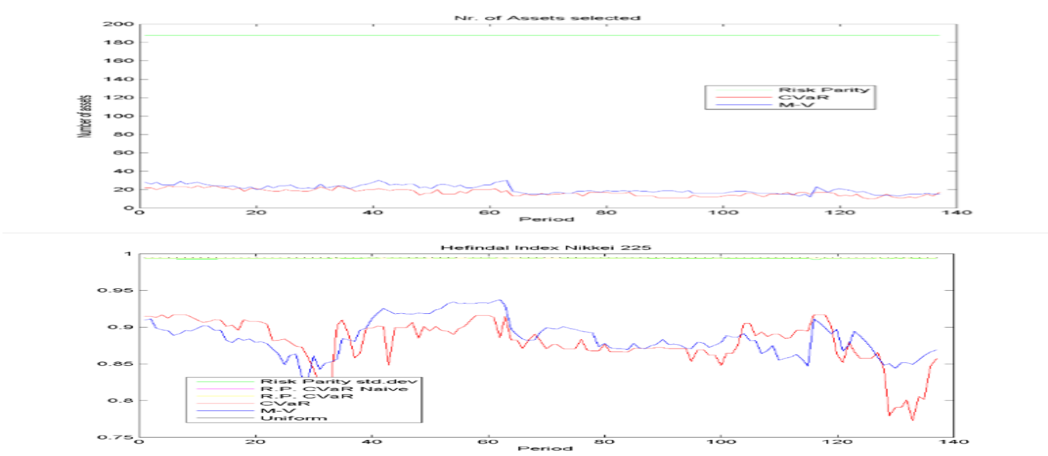

Figure 2. The number of assets selected from the possible 188 from each model (top chart) and the Herfindal index (bottom chart)

The Uniform and Risk Parity strategies exceed the threshold for all 188 assets, so the graph is omitted. Mean-Variance and Expected Shortfall focus on a small group; on this subset, we apply the Risk Parity strategies.

Before proceeding, we examine diversification using the Herfindahl index: On the subset selected first with Mean-Variance and then with Expected Shortfall, we apply the Risk Parity strategies. The results are in reported in Table 3.

Table 3. The performance of the portfolios with assets of NIKKEI225 on the second selection

	R.P. STD	M-V	R.P. ES.N	R.P. ES	$\text{ES}_{10\%}(x)$	Uniform
$\mu(\%)$	0.0158	0.0031	0.0377	0.0324	0.0272	0.0449
$\mu_{\text{ann}}(\%)$	0.823	0.1628	1.9812	1.6973	1.4255	2.3621
$\mu^c(\%)$	-7.38	-13.66	4.2832	1.0310	-0.471	-4.359
$\sigma(\%)$	2.3748	2.3808	2.3865	2.3974	2.3160	3.1991
S_σ	0.0481	0.0095	0.1151	0.0982	0.0854	0.1024

Source: Computation in Matlab

The second table reflects a more stable but less profitable environment compared to the first. Average and annualized returns are generally lower across most strategies, except "Uniform", which maintains the highest returns in both cases. Risk Parity (RP) strategies exhibit reduced volatility in the second table, suggesting enhanced stability, though at the cost of lower returns.

Expected Shortfall (ES) strategies show improved risk-adjusted performance, as indicated by higher Sharpe ratios. Mean-Variance (M-V) continues to underperform, with low returns and poor risk-adjusted metrics.

While Uniform consistently delivers the highest returns, its high volatility and moderate Sharpe ratio suggest a need for better risk management. Overall, risk-managed strategies like RP ES and ES are more suitable for investors prioritizing stability and risk control.

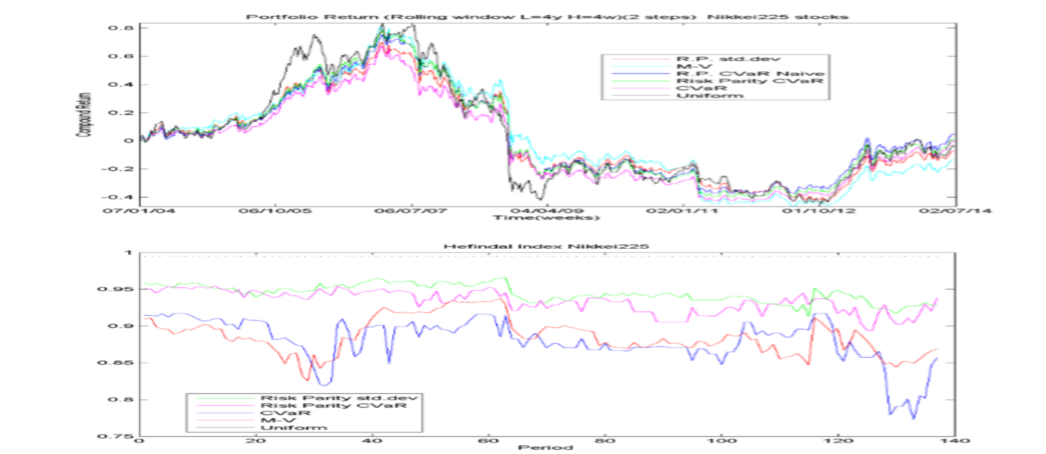

Figure 3. The performance out of sample for Nikkei225 (188 stocks) with the second selection (top chart) and the Herfindal index for diversifications (bottom chart)

The graph shows the compound return of portfolio strategies applied to Nikkei 225 stocks over time, using a rolling window of 4 years and step size of 4 weeks. From 2004 to 2007, all strategies experience consistent growth. The 2008 crisis causes a sharp decline in returns across all strategies. Post-crisis, returns stabilize and recover around 2012. CVaR and Risk Parity strategies show more stable recoveries, highlighting robustness during market turbulence.

The Herfindahl Index measures market concentration. A higher value indicates concentration; a lower value suggests greater diversification.

Risk-managed strategies (e.g., CVaR, Risk Parity CVaR) perform better during volatile periods by limiting drawdowns. Uniform achieves the highest returns during bull markets but is highly susceptible to losses during downturns, making it suitable for aggressive investors.

3.2 Case 2: Smaller set of mixed assets

In Table 4, a collection of twenty financial assets is presented, organized into three main categories: stocks, fixed-income securities, and cryptocurrencies. This classification contributes to comprehending the asset composition for portfolio analysis, especially in research investigating diversification and risk optimization among various asset classes. The time series utilized covers from March 19, 2024, to March 18, 2025. Concerning the stocks and bonds, we have 251 observations as they represent the prices at the adjusted closure. These three categories have distinct trading days; for instance, the cryptocurrency market is always open for trading. The time series is divided in two parts. The first part is used to estimate the optimal weights for each portfolio and is called the in-sample period L, which is 125 days of observations. Then, starting from October 1, the performance of the portfolio is calculated in terms of compound returns for 5 days, referred to as the holding period h. After that, for the calculation, it is necessary to roll these two periods each day until the last week.

Table 4. List of Assets of a Mixed Portfolio

Asset	**Category**
1. Tesla, Inc. Common Stock (TSLA)	Stock
2. Apple Inc. Common Stock (AAPL)	Stock
3. Meta Platforms, Inc. Class A Common Stock (META)	Stock
4. NVIDIA Corporation Common Stock (NVDA)	Stock
5. Amazon.com, Inc. Common Stock (AMZN)	Stock
6. Broadcom Inc. Common Stock (AVGO)	Stock
7. Baidu, Inc. ADS (BIDU)	Stock
8. Advanced Micro Devices, Inc. Common Stock (AMD)	Stock
9. Alphabet Inc. Class A Common Stock (GOOGL)	Stock
10. Microsoft Corporation Common Stock (MSFT)	Stock
11. 1 Yr Constant Maturity Treasury (CMTN1Y)	Fixed Income
12. JPMorgan Government Bond Fund A (OGGAX)	Fixed Income
13. Vanguard Emerging Markets Government Bond ETF (VWOB)	Fixed Income
14. American Century Government Bond Fd A Cl (ABTAX)	Fixed Income
15. Bitcoin (BTC)	Cryptocurrency
16. Ethereum (ETH)	Cryptocurrency
17. Litecoin (LTC)	Cryptocurrency
18. Cardano (ADA)	Cryptocurrency
19. Stellar (XLM)	Cryptocurrency
20. Dogecoin (DOGE)	Cryptocurrency

Source: Nasdaq.com

This mixed portfolio consists of 60% stocks, 20% fixed income, and 20% cryptocurrencies. The equity portion includes high-growth tech stocks like Tesla, Apple, NVIDIA, and Microsoft, offering strong potential for capital appreciation but with higher volatility. The fixed-income segment (20%) provides stability through government bonds and bond funds, reducing overall risk. The cryptocurrency allocation (20%) adds high-risk, high-reward exposure with assets like Bitcoin and Ethereum, introducing diversification beyond traditional markets.

Overall, this portfolio balances growth, income, and speculative assets, catering to a moderately aggressive investor comfortable with market fluctuations. This portfolio is well-suited for an investor with a medium-to-high risk tolerance who seeks long-term growth while maintaining some downside protection. The tech-heavy stock selection leverages innovation and market dominance but may be sensitive to economic cycles. The fixed-income allocation acts as a buffer during market downturns, providing steady returns. The crypto exposure introduces volatility but also potential for outsized gains, appealing to those bullish on digital assets. Diversification across these three asset classes helps mitigate risk while positioning for growth in both traditional and emerging markets.

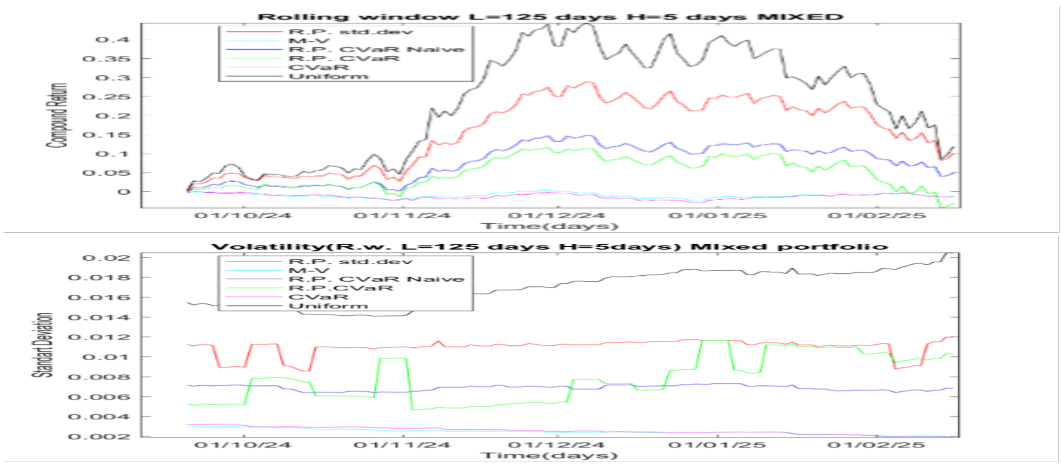

Figure 4. The out-of-sample performance of the portfolio mixed with stocks, bonds and cryptocurrencies (top chart) and the out-of-sample volatility of the portfolio mixed with stocks, bonds and cryptocurrencies (bottom chart)

Fig. 4, top chart, illustrates the compounded returns of the portfolios over the out-of-sample period. The results likely show that portfolios including cryptocurrencies exhibit higher returns compared to traditional portfolios, reflecting the high-risk, high-reward nature of crypto-assets. However, the volatility of these returns should be noted, as cryptocurrencies are known for their price swings. The comparison between different optimization models (e.g., Min Variance, Risk Parity) would highlight which strategy balances returns and risk most effectively.

The portfolios' volatility (standard deviation) is shown in Figure 8. It is anticipated that the intrinsic price volatility of cryptocurrencies will lead to a rise in the overall volatility of investment portfolios. The Min Variance model is expected to demonstrate the lowest volatility, whereas the naïve $1/N$ or crypto-heavy portfolios may show the highest. A major point to note is the trade-off between returns and volatility.

Cryptocurrencies might improve diversification due to their low correlation with traditional assets, while optimized portfolios (e.g., Min Variance) may concentrate weights in fewer assets.

In Fig. 5–middle chart– when we apply the two-step selection, we notice that most of the dominance order is maintained as in the previous case, except that Risk Parity strategies with CVaR change order. This result shows that the subset selection achieves better improvement if the set is larger.

While the CVaR and Mean Variance show the lowest volatility, the Risk Parity with CVaR performs better in terms of riskiness compared to the remaining models.

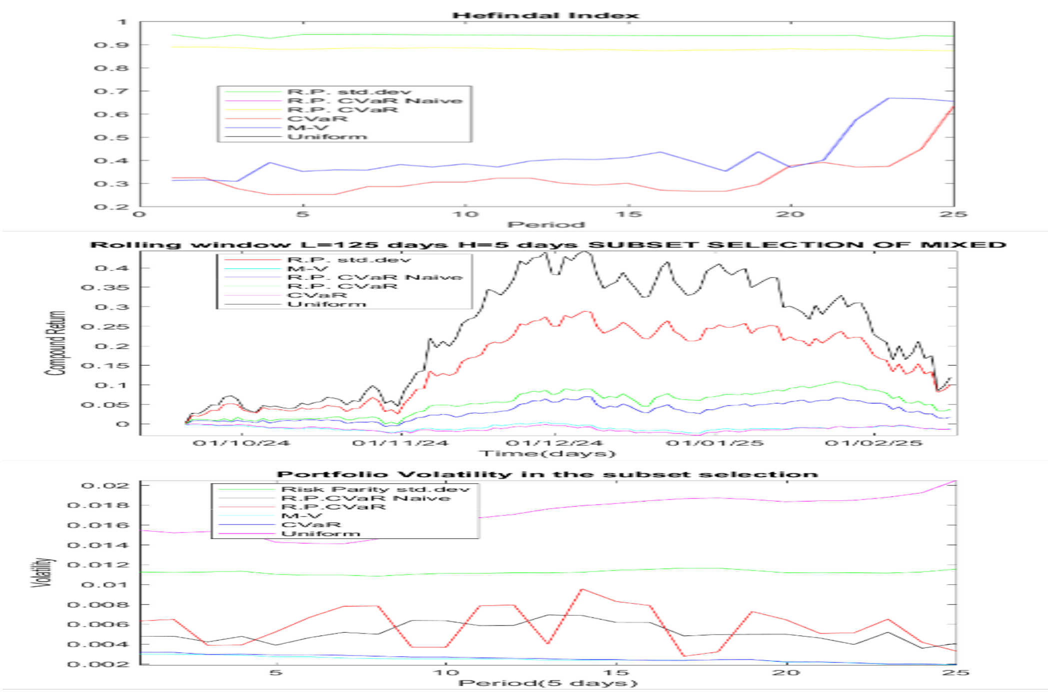

Figure 5. The Herfindal Index (top chart), the out-of-sample performance of the portfolio mixed with stocks, bonds and cryptocurrencies (middle chart), and the out-of-sample volatility of the portfolio mixed with stocks, bonds and cryptocurrencies for the subset selection (bottom chart)

4 Conclusion

This study presents a comprehensive framework that addresses critical limitations in modern portfolio optimization by integrating rigorous asset selection with advanced risk allocation techniques. Our two-stage approach, combining Mean-Variance and Expected Shortfall filtering with modified Risk Parity implementation, demonstrates significant improvements in risk-adjusted performance across diverse market conditions. The empirical results, drawn from both traditional equity markets (Nikkei 225) and mixed-asset portfolios including cryptocurrencies, reveal several key insights that advance both academic discourse and practical portfolio management. The findings robustly confirm that while naive diversification strategies may achieve higher nominal returns, they do so at the cost of excessive volatility and severe drawdowns during market crises, a conclusion that aligns with and extends the foundational work of Markowitz [1952] and subsequent risk management literature Artzner and Delbaen [1999], Rockafellar and Uryasev [2000]. Our methodology specifically addresses the asset selection gap in Risk Parity approaches identified by Qian [2005] and Maillard et al. [2009], demonstrating that purposeful pre-selection of assets enhances the effectiveness of subsequent risk balancing.

Several theoretical and practical contributions emerge from this research:

1. **Integration of Risk Measures:** By employing Expected Shortfall as both a filtering criterion and allocation metric, we operationalize the theoretical advances of Tasche

[2000] and Stefanovits [2010], proving their practical value in comprehensive portfolio construction.

2. **Adaptability to Market Conditions:** The framework's strong performance during the 2008 crisis period and in cryptocurrency-inclusive portfolios Platanakis and Urquhart [2020], Veliu and Aranitasi [2024] demonstrates its robustness across different market regimes and asset classes.

3. **Computational Efficiency:** Building on the optimization techniques of Bjerring et al. [2016], we present a methodology that maintains mathematical rigor while remaining computationally feasible for real-world implementation.

The practical implications for portfolio managers are substantial. Institutional investors can apply this framework to:

- Enhance pension fund allocations by improving risk-adjusted returns
- Construct more resilient sovereign wealth portfolios
- Develop next-generation risk parity funds with better asset selection protocols

Future research directions could explore:

1. Extension to additional alternative assets (private equity, real estate)
2. Incorporation of machine learning techniques for dynamic threshold adjustment
3. Application to environmental, social, and governance (ESG) constrained portfolios

As financial markets continue evolving with technological innovations and regulatory changes, the need for adaptive, robust portfolio strategies becomes increasingly critical. This research provides a systematic approach that balances theoretical sophistication with practical implementation, offering investors a powerful tool to navigate the complexities of modern portfolio management while maintaining disciplined risk controls. The framework's success in both traditional and cryptocurrency-enhanced portfolios suggests its potential as a unifying methodology for 21st-century asset allocation, one that respects the insights of classical portfolio theory while embracing the challenges and opportunities of contemporary financial markets. By demonstrating how thoughtful integration of asset selection and risk allocation can create more resilient portfolios, this study makes a significant contribution to the ongoing development of portfolio management practices.

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