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THE UNIFED THEORY OF DEBRIS FLOW INITIATION BY USING HOMOGENIZATION THEORY

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ABSTRACT

We attempt to find the unified theory for the prediction of the initiation of debris-flow by using homogenization theory. In this study, we show the leading order solution, which is the first step of this derivation of unified theory. The derivation started in the microscopic scale in the soil. The representative elementary volume (REV) in the soil is set to be one order larger than the scale of porosity. Solids in the REV are assumed to be rigid and adhesion-less. The liquid velocity in the porosity is slow. By the no-slip boundary condition and periodicity of REV, we could obtain the microscopic flow conditions. Using the assemble average with time dependence taken into account, we obtain the macroscopic relation of water content with the spatial and time variables from the microscopic flow conditions. This macroscopic equation could be validated by the Richards' equation.

***Key words**: homogenization theory, Richards' equation*

INTRODUCTION

It is well accepted concept that landslide together with enough water can produce debris flows. But the mechanism for landslides and occurrence of debris flows are different. If we consider these as continuum, physics involved is different. Landslide is a bulky motion of a soil where particle displacement is important. However, debris flow is a flowing process where strain rate is important. This means as the motion change from landslides to debris flows, the controlling physics also changed. But if we consider these phenomena from scales of particles, the major difference would be the interactions between solids and fluid motion in the pore scale.

From the continuum point of view, there are many theories used to interpreting the flowing properties of debris flow. An extensive review was given by ANCEY (2007). Many theories are validated useful and practical in certain domains. However, most of these theories can be used either when bulk material is almost stationary (such a soil) or has large movement (such a debris flows and avalanches). IVERSON *et alii* (1997) gave a review for models involving the effect of pore pressures and granular temperature in the mobilization of debris-flow. In the same paper, they assessed the relationship between Coulomb failure and liquefaction, and considered the role of granular temperature and soil volume change in an infinite-slope formulation. IVERSON(2000) also proposed multiple time-scales together with Richards' equation to develop a mathematical model to evaluate effects of rainfall infiltration on landslide occurrence, depth, and acceleration. The model provided a tool to assess the possibility of landslide triggered by rainfall and post-failure motion. But this approach still used the continuum concept to model landslide process macroscopically.

As the continuum motion is actually the result from small scale motion, there should be a method to examine the small scale motion and then transfer motion of these small scale to that of continuum. In such small scale, one should be able to visualize how particles start from stationary and then change to collision based motion. As the first attempt, we shall use this approach to examine if the well known equation such as Richard's equation which is based on experimental results can be derived theoretically.

Therefore, we propose a new way to study the initiation process. The initiation process starts from static solids and flowing liquid in the pore. Then gradually it develops to solid movement with strong interaction of soil and liquid in the pore scale as well as bulk motion of solid-liquid mixture. To study the phenomenon, two drastically different concepts must be used. Interaction between liquid and individual solids is usually considered with Lagrangian coordinates and bulk motion is usually considered with Eulerian coordinates. In order to combine these two, there must be two or more different characteristic length scales involved in this initiation process. It is reasonable to believe that the initiation of appreciable solid velocity has something to do with effects from different scales. Homogenization theory (Auriault, 1991) has been applied in this aspect and successfully derived the flow condition of seepage in the pore under saturated and static soil. Therefore, we shall adopt similar approach to study the initiation process. In this study, we show the leading order solution which is the first step towards our goal.

Without any assumption of constitutive law of the water-soil mixture, we begin to derive seepage flow condition in the representative element volume in the microscopic length scale -- the scale of the order of pore. Then we use assemble average to obtain the averaged flow condition in the macroscopic scale -- the scale of total bulk soil-water mixture. In the end, we can obtain the same result as Richards' equation (Richards, 1931).

FUNDAMENTALS OF HOMOGENIZATION THEORY

Homogenization theory is a method to obtain a motion equation of interest by using a multiple-scale perturbation method together with assembled averages in smaller scales.

The first step for homogenization is to decide the two different characteristic length scales, micro- and macro-scale which represent the characteristic length scale in the pore level and under outer physical mechanism respectively. Using these two scales, we could define the small parameter as below.

$$\varepsilon = \frac{l}{L} \ll 1 \tag{1}$$

where l and L are micro- and macro-scopic characteristic length scales respectively. The volume in micro-scale is called representative element volume (REV from here on). By this small parameter, we also define the multiple-scale spatial and temporary independent variables as

$$x_i = (x_{0i}, x_{1i}, x_{2i}, \ldots) \text{ and } t = (t_0, t_1, t_2, \ldots) \tag{2}$$

where $x_{1i} = \varepsilon x_{0i}$,$x_{2i} = x_{0i}$, ... and $t_1 = \varepsilon t_0$, $t_2 = \varepsilon t_0$, ... and so on. The physical dependent variable $\Phi(x_i, t)$represents velocity, pressure or other perturbed physical variables in later derivation. They are expanded by the small parameter in (1) as follow

$$\Phi(x_i, t) = \Phi_0(x_i, t) + \varepsilon \Phi_1(x_i, t) + \varepsilon^2 \Phi_2(x_i, t) + \ldots \tag{3}$$

In (3), for k^{th}-order term, the $(k+1)^{th}$ or higher order termspossess the property of periodicity in k-th order REV. A compatibility condition exists between equations of different order so as to assure the solutions in different orders are independent. The condition (Auriault, 1991) is

$$\frac{\partial \Phi_{(i+1)}}{\partial x_{(i)}} + \frac{\partial \Phi_{(i)}}{\partial x_{(i+1)}} = 0 \tag{4}$$

Substituting (2) and (3) into the governing equations of our problem, we could solve the micro-scopic solution with the boundary conditions and compatibility condition. Then, we use the spatial assemble average in the REV to obtain the averaged physical variable representing the macro-scopic property. The assemble average is defined as

$$\left\langle \Phi_{(k)} \right\rangle = \frac{1}{\left|\Omega_{(k)}\right|} \int_{\Omega_{(k)}} \Phi_{(k)} d\Omega_{(k)} \tag{5}$$

where $|\Omega_{(k)}|$is total volume of k^{th}-order REV. $\Phi_{(k)}$is the assemble averaged of Φ(k) in the kth-order REV, representing the $(k+1)^{th}$-order property and becomes the function of macroscopic (higher order) independent variables $x_{(k+1)i}$, $x_{(k+2)i}$ ….etc. The volume $\Omega_{(k)}$ can also be a function of temporary independent variable in the unsaturated soil. Then the physical properties of the bulk solid-liquid mixture can be found using these averaged results with boundary conditions under marco-scopic scale.

GOVERNING EQUATIONS

In this study, our problem is to derive the seepage flow condition in unsaturated and static soil. We consider the pores in soil are large enough for water to form a free surface interface of airliquid (Fig. 1).

If solid structure is stationary, we only have the governing equations for pore water, which is the Navier-Stokes equations

$$\frac{\partial u_i}{\partial x_i} = 0 \qquad (6)$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} \qquad (7)$$

where ρ and μ are the density and dynamic viscosity of water. $p = p' + \rho gz$ is pressure with p' being dynamic pressure. For boundary conditions, we need them for different scales. At microscopic scale, there are kinematic boundary conditions at the air-liquid and solid-liquid interface. There is dynamic boundary condition at free surface. At macroscopic scale, we use periodicity of REV. These conditions are listed below.

$$u_i = 0 \text{ at } x_i \in \Gamma \qquad (8)$$

$$\frac{DH}{Dt} = 0 \text{ at free surface} \qquad (9)$$

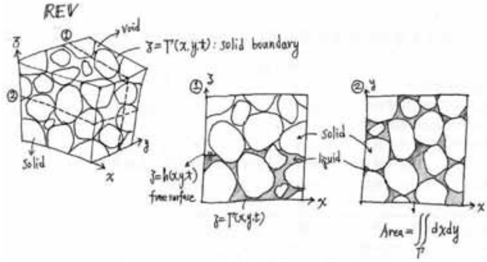

Fig.1 - The definition of free surface of water and solid boundary in the REV

$$\left[\left(-p + \delta_s \kappa\right)\delta_{ij} + 2\mu e_{ij}\right] \cdot \frac{\partial H}{\partial x_i} = 0 \text{ at free surface} \qquad (10)$$

u_i is periodicity in micro-scale (11)

where Γ is the solid boundary in REV, H the free surface of water in the unsaturated REV, that is H =, z- h(x, y, t) ,δ_s the surfacetension coefficient ranging from 0.019 at 0°C to 0 at 100°C (White, 2006). k is the curvature of the free surface. δ_{ij} and e_{ij} are Kronecker delta the strain-rate tensor respectively. (8) is no-slip condition on the solid surface.. Boundary conditions, (9) and (10), would be only used in the unsaturated micro-scopic REV. (11) is the condition of periodicity in the saturated REV

From the definition, eq.(5), if we want to obtain the averaged seepage flow condition, u should be averaged as in (5), so.

$$\left\langle u_{(k)i} \right\rangle = \frac{1}{\left|\Omega_{(k)}\right|}\int_{\Omega_{l(k)}} u_{(k)i}\, d\Omega_{(k)} \qquad (12)$$

where $\Omega_{l(k)}$ is the liquid volume in k^{th}-order REV. We also define the porosity η and water content θ in k^{th}-order as

$$\eta_{(k)} = 1 - \frac{\Omega_{s(k)}}{\left|\Omega_{(k)}\right|} \qquad (13)$$

$$\theta_{(k)} = \frac{\Omega_{l(k)}}{\left|\Omega_{(k)}\right|} \qquad (14)$$

where $\Omega_{s(k)}$ is the total solid volume in in k^{th}-order REV, and $\eta_{(k)}$ and θ(k) are all the function of $x_{(k+1)i}$, $x_{(k+2)i}$,...,$t0$,$t1$,... . If the k^{th}order REV is saturated, we could have the relation that $|\Omega(k)| = \Omega_{l(k)} + \Omega_{s(k)}$. But in unsaturated REV, $\Omega_{l(k)}$ can vary in time. In most soil. Water content θ ranges from 0 to η , and the porosity η ranges from 0.25 to 0.75 (Chow *et alii*, 1988)

NORMALIZATION

We define the microscopic characteristic length, l , is one order larger than the characteristic length of pores in soil; the macroscopic length, L , is the outer characteristic length of all bulk. Using these two scales, the small parameter ε = l /L can be defined. In our problem, the outer physical excitation in the microscopic REV is the macroscopic pressure gradient. Due to the viscous effects dominate in the flow in pores, we assume that the viscous term in micro-scale is as important as the macro-scale pressure gradient. So all scales are defined.

$$\left(x_i, p, u_i, t, \mathrm{H}\right) = \left(l, p, \frac{pl^2}{\mu L}, \frac{\mu L}{pl}, l\right)\left(x_i', p', u_i', t', \mathrm{H}\right) \tag{15}$$

Substituting all scales in (15) to equation (6) to (11), and omitting the primes, we obtain the dimensionless equations as

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{16}$$

$$\varepsilon \,\mathrm{Re}\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \varepsilon \frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{17}$$

With normalized boundary conditions.

$$u_i = 0 \ \text{ at } x_i \in \Gamma \tag{18}$$

u_i is periodicity in micro-scale (19)

$$\frac{D\mathrm{H}}{Dt} = 0 \tag{20}$$

$$\left[\left(-p + \beta\kappa\right)\delta_{ij} + 2e_{ij}\right] \cdot \frac{\partial \mathrm{H}}{\partial x_i} = 0$$

$$\mathrm{Re} = \frac{\varepsilon \rho l^2 P}{\mu^2} \ , \ \beta = \frac{\delta_s P}{\mu L} \ \text{and} \ e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{21}$$

Re is the Reynolds number of seepage flow in pores. From the expression it is of O(ε) in our problem; β is the ratio of the effect fo surface tension to shear stress at free surface in the pores within εunsaturated REV.

Using small parameter ε , we expand the velocity, pressure of water and free surface

$$u_i = u_{0i} + \varepsilon u_{1i} + \varepsilon^2 u_{2i} + \dots \tag{22}$$

$$p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \dots \tag{23}$$

$$\mathrm{H} = \mathrm{H}_0 + \varepsilon \mathrm{H}_1 + \varepsilon^2 \mathrm{H}_2 + \dots \tag{24}$$

Substituting (22) to (24) into eq. (16) to (21) and collecting terms of the same order, we obtain

$O(\varepsilon^0)$:

$$\frac{\partial u_{0i}}{\partial x_{0i}} = 0 \ \text{ and } \ 0 = -\frac{\partial p_0}{\partial x_{0i}} \tag{25 a b}$$

$O(\varepsilon^1)$:

$$\frac{\partial u_{1i}}{\partial x_{0i}} + \frac{\partial u_{0i}}{\partial x_{1i}} = 0 \ \text{ and } \ 0 = -\left(\frac{\partial p_1}{\partial x_{0i}} + \frac{\partial p_0}{\partial x_{1i}}\right) + \frac{\partial^2 u_{0i}}{\partial x_{0j}\partial x_{0j}} \tag{26 a b}$$

$O(\varepsilon^2)$:

$$\frac{\partial u_{2i}}{\partial x_{0i}} + \frac{\partial u_{1i}}{\partial x_{1i}} + \frac{\partial u_{0i}}{\partial x_{2i}} = 0 \tag{27a}$$

$$\frac{Du_{0i}}{Dt_0} = -\left(\frac{\partial p_2}{\partial x_{0i}} + \frac{\partial p_1}{\partial x_{1i}} + \frac{\partial p_0}{\partial x_{2i}}\right) + \frac{\partial^2 u_{0i}}{\partial x_{1j}\partial x_{0j}} + \frac{\partial^2 u_{1i}}{\partial x_{0j}\partial x_{0j}} \tag{27b}$$

$$\text{Where } \frac{D}{Dt_0} = \frac{\partial}{\partial t_0} + u_{0j}\frac{\partial}{\partial x_{0j}}$$

Eq (25a), (26a) and (27a) are continuity in different orders of ε (25b), (26b) and (27b) are momentum equations tensor. Besides, the boundary conditions of k^{th}-order of ε in saturated soil are

$$u_{ki} = 0 \ \text{ at } x_i \in \Gamma \tag{28}$$

u_{ki} is periodicity in micro-scale (29)

$O(\varepsilon^0)$:

$$\frac{\partial \mathrm{H}_0}{\partial t_0} + u_{0j}\frac{\partial \mathrm{H}_0}{\partial x_{0j}} = 0 \tag{30 a b}$$

$$\left(-p_0 + \beta\frac{\partial^2 \mathrm{H}_0}{\partial x_{0i}{}^2} + \left(e_{ij}\right)_{0,0}\right)\frac{\partial \mathrm{H}_0}{\partial x_{0i}} = 0$$

$O(\varepsilon^1)$:

$$\frac{\partial \mathrm{H}_1}{\partial t_0} + u_{0j}\frac{\partial \mathrm{H}_1}{\partial x_{0j}} + \frac{\partial \mathrm{H}_0}{\partial t_1} + u_{1j}\frac{\partial \mathrm{H}_0}{\partial x_{0j}} + u_{0j}\frac{\partial \mathrm{H}_0}{\partial x_{1j}} = 0$$

$$\left[-p_1 + \beta\left(\frac{\partial^2 \mathrm{H}_1}{\partial x_{0i}{}^2} + \frac{\partial^2 \mathrm{H}_0}{\partial x_{1i}\partial x_{0i}}\right) + \left(e_{ij}\right)_{1,0} + \left(e_{ij}\right)_{0,1}\right]\frac{\partial \mathrm{H}_0}{\partial x_{0i}} \tag{31 a b}$$

$$+\left[-p_0 + \beta\frac{\partial^2 \mathrm{H}_0}{\partial x_{0i}{}^2} + \left(e_{ij}\right)_{0,0}\right]\cdot\left(\frac{\partial \mathrm{H}_1}{\partial x_{0i}} + \frac{\partial \mathrm{H}_0}{\partial x_{1i}}\right) = 0$$

$O(\varepsilon^2)$:

$$\frac{\partial \mathrm{H}_2}{\partial t_0} + \frac{\partial \mathrm{H}_1}{\partial t_1} + \frac{\partial \mathrm{H}_0}{\partial t_2} + u_{2j}\frac{\partial \mathrm{H}_0}{\partial x_{0j}} + u_{0j}\frac{\partial \mathrm{H}_0}{\partial x_{2j}} + u_{1j}\frac{\partial \mathrm{H}_0}{\partial x_{1j}} \tag{32 a}$$

$$+u_{1j}\frac{\partial \mathrm{H}_1}{\partial x_{0j}} + u_{0j}\frac{\partial \mathrm{H}_1}{\partial x_{1j}} + u_{0j}\frac{\partial \mathrm{H}_2}{\partial x_{0j}} = 0$$

$$\left[-p_2 + \beta\left(\frac{\partial^2 \mathrm{H}_2}{\partial x_{0i}{}^2} + \frac{\partial^2 \mathrm{H}_1}{\partial x_{1i}\partial x_{0i}} + \frac{\partial^2 \mathrm{H}_0}{\partial x_{1i}{}^2}\right) + \left(e_{ij}\right)_{2,0} + \left(e_{ij}\right)_{1,1} + \left(e_{ij}\right)_{0,2}\right]\cdot\frac{\partial \mathrm{H}_0}{\partial x_{0i}}$$

$$+\left(-p_1 + \beta\left(\frac{\partial^2 \mathrm{H}_1}{\partial x_{0i}{}^2} + \frac{\partial^2 \mathrm{H}_0}{\partial x_{1i}\partial x_{0i}}\right) + \left(e_{ij}\right)_{1,0} + \left(e_{ij}\right)_{0,1}\right)\cdot\left(\frac{\partial \mathrm{H}_1}{\partial x_{0i}} + \frac{\partial \mathrm{H}_0}{\partial x_{1i}}\right) \tag{32 b}$$

$$+\left[-p_0 + \beta\frac{\partial \mathrm{H}_0}{\partial x_{0j}{}^2} + \left(e_{ij}\right)_{0,0}\right]\cdot\left(\frac{\partial \mathrm{H}_2}{\partial x_{0i}} + \frac{\partial \mathrm{H}_1}{\partial x_{1i}} + \frac{\partial \mathrm{H}_0}{\partial x_{2i}}\right) = 0$$

$$\text{Where } \left(e_{ij}\right)_{m,n} = \frac{\partial u_{mi}}{\partial x_{nj}} + \frac{\partial u_{ni}}{\partial x_{mj}} \tag{32 c}$$

Eq (25) to (32) are all the equations and boundary conditions in our problem. In the following, we begin to derive the flow condition in unsaturated soil.

DERIVATION OF UNSTEADY FLOW IN UNSATURATED SOIL

In the bulk of solid-liquid mixture, there must exist saturated and unsaturated REV. We firstly derive the flow condition insaturated REV, and then continue to derive flow in unsaturated REV.From (25b), we find

$$p_0 = p_0(x_{1i}, x_{2i}, ..., t) \quad (33)$$

This implies 0th-order pressure depends on macroscopic variables for 0th-order REV. To solve u_{0i} , we combine (33) and (25b) to get

$$\frac{\partial p_0}{\partial x_{1i}} = -\frac{\partial p_1}{\partial x_{0i}} + \frac{\partial^2 u_{0i}}{\partial x_{0j} \partial x_{0j}} \quad (34)$$

Due to the linearity of (34), the solution form of u_{0i} and $p1$ are (MEI & AURIAULT, 1991)

$$u_{0i} = K_{ij} \frac{\partial p_0}{\partial x_{1j}} \text{ and } p_1 = A_j \frac{\partial p_0}{\partial x_j} + \overline{p}_1 \quad (35\text{ a b})$$

where *Kij* and *Aj* are the 2nd and 1st-order tensors representing the geometrical properties in the 0th-order REV. $\overline{p1}$ is the function of x_{1i} , x_{2i} ,...,t , and is a constant representing outer physical excitation for 0th-order pressure. Applying (35a,b) with (25a) and (36b), we obtain

$$\frac{\partial K_{ij}}{\partial x_{0i}} = 0 \quad (36)$$

$$1 = -\frac{\partial A_j}{\partial x_{0i}} + \frac{\partial^2 K_{ij}}{\partial x_{0k} \partial x_{0k}} \quad (37)$$

and boundary conditions for water in saturated REV become

$$K_{ij} = 0 \ \text{ at } x_i \in \Gamma \quad (38)$$

$K_{ij} \frac{\partial p_0}{\partial x_{1j}}$ is periodicity in micro-scale

For unsaturated REVs, the free surface boundary conditions become

$$\frac{\partial \mathrm{H}_0}{\partial t_0} + K_{jk} \frac{\partial p_0}{\partial x_{1k}} \frac{\partial \mathrm{H}_0}{\partial x_{0j}} = 0 \quad (40)$$

$$\left(-p_0 + \beta \frac{\partial^2 \mathrm{H}_0}{\partial {x_{0i}}^2}\right) \frac{\partial \mathrm{H}_0}{\partial x_{0i}} + \left(\frac{\partial K_{jk}}{\partial x_{0i}} + \frac{\partial K_{ik}}{\partial x_{0j}}\right) \frac{\partial p_0}{\partial x_{1k}} \cdot \frac{\partial \mathrm{H}_0}{\partial x_{0i}} = 0 \quad (41)$$

Equations (36) to (39) are a boundary-value-problem for solving K_{ij} and A_j in saturated REV. For specific sample of soil, it is possible to define Γ and then solve the whole set of equations.

However, it is difficult to define the solid boundary Γ in the microscopic REV in general without any knowledge for the soil. Due to the complex composition of heterogeneous materials in nature, one often obtain K_{ij} and A_j through experimental methods.

VERIFICATION WITH RICHARD'S EQUATION

In this paper, we do not need the detail of K_{ij} and A_j to verify our theory. We shall show that our theory is equivalent to Richard's equation with the same assumption, i.e. isotropic and homogeneous

Substituting (35a) into (26a) and applying assemble average in a saturated REV, we obtain

$$\left\langle \frac{\partial u_{1i}}{\partial x_{0i}} \right\rangle + \frac{\partial}{\partial x_{1i}} \left(\left\langle K_{ij} \right\rangle \frac{\partial p_0}{\partial x_{1i}} \right) = 0 \quad (42)$$

With periodic condition of u_{1i} in 0th-order REV, we could eliminate the first term in LHS of (42). So

$$\frac{\partial}{\partial x_{1i}} \left(\left\langle K_{ij} \right\rangle \frac{\partial p_0}{\partial x_{1i}} \right) = 0 \quad (43)$$

where K_{ij} is averaged hydraulic conductivity in microscopic REV, and it can be regarded as the representing hydraulic conductivity in macro-scale. If the soil is isotropic and homogenous, we could simplify K_{ij} t o K_{ij} , where K is a constant, and in this case. (43) becomes

$$\frac{\partial^2 p_0}{\partial x_{1i} \partial x_{1i}} = 0 \quad (44)$$

By applying the macroscopic boundary conditions to (44), we get the pressure distribution $p0$ in saturated soil. Furthermore, taking this solved $p0$ back to (35a), we obtain velocity of seepage in the micro-scale. The result is the same as MEI & AURIAULT(1991).

For unsaturated REV, we apply assemble average to (25a)

$$\left\langle \frac{\partial u_{0i}}{\partial x_{0i}} \right\rangle = \frac{1}{|\Omega_0|}\int_{\Omega_{l0}} \frac{\partial u_{0i}}{\partial x_{0i}} d\Omega_0 = 0 \quad (45)$$

With Divergence theorem, we obtain

$$0=\left\langle \frac{\partial u_{0i}}{\partial x_{0i}} \right\rangle = \frac{1}{|\Omega_0|}\int_s u_{0i}\cdot n_i dS_0 \quad (46)$$

where S is the interface of water in 0th-order REV. There are three different kinds of interfaces, water-solid interface S_Γ , water-air interface S_H , and water area on each surface of a REV SREV . Separate (46) for different kinds of interface, we obtain

$$0=\int_s u_{0i}\cdot n_i dS_0 = \int_{S_H} u_{0i}\cdot n_i dS_0 + \int_{S_{REV}} u_{0i}\cdot n_i dS_0 + \int_{S_\Gamma} u_{0i}\cdot n_i dS_0$$

where n_i is an unit normal vector of each surface, and S0 is the element surface of 0th-order REV in integral. With no-slip condition, the last term on RHS of (46) is zero.

Dividing $|\nabla H_0|$ from the free-surface kinematic boundary condition (40), we obtain

$$\frac{1}{|\nabla H_0|}\frac{\partial H_0}{\partial t_0} + K_{jk}\frac{\partial p_0}{\partial x_{1k}}\cdot n_j = 0 \quad (47)$$

$$\text{where } n_j = \frac{1}{|\nabla_0 H_0|}\frac{\partial H_0}{\partial x_{0j}} \quad (48)$$

the subscript zero of the $\nabla_0 H_0$ means taking gradient with 0th-order spatial independent variables, x_0, y_0 and z_0 . In (47), the second term is just u_{0i} - n_i . So this is simply the normal flux at free surface of water.

$$u_{0i}\cdot n_i = -\frac{1}{|\nabla_0 H_0|}\frac{\partial H_0}{\partial t_0} \quad (49)$$

Finally, the second term in RHS of (46) could be expressed by following equation.

$$\int_{S_{REV}} u_{0i}\cdot n_i dS_0 = \int_{S_x} u_{0x}\cdot n_x dS_0 + \int_{S_y} u_{0y}\cdot n_y dS_0 + \int_{S_z} u_{0z}\cdot n_z dS_0 \quad (50)$$

where S_x , S_y and S_z are the areas of the surface normal to yz-, xz and zy-plane respectively of a REV. Taking (35a) into (50) together with each unit normal vectors of its surface of REV and rearranging it, we obtain

$$\int_{S_{REV}} u_{0i}\cdot n_i dS_0 = \frac{\partial p_0}{\partial x_{1j}}\left(\int_{S_x} K_{xj}dS_0 + \int_{S_y} K_{yj}dS_0 + \int_{S_z} K_{zj}dS_0\right) \quad (51)$$

We define the terms in brackets in RHS of (51) as below.

$$\overline{K}_j = \int_{S_x} K_{xj}dS_0 + \int_{S_y} K_{yj}dS_0 + \int_{S_z} K_{zj}dS_0 = \int_{S_k} K_{kj}dS_0 \quad (52)$$

where $\overline{K}_j$ is the hydraulic conductivity of REV in each direction of x, y and z. Then, (51) could be changed into the form as below.

$$\int_{S_{REV}} u_{0i}\cdot n_i dS_0 = \frac{\overline{K}_j}{|\nabla_0 H_0|}\frac{\partial p_0}{\partial x_{1j}} \quad (52\text{ b})$$

Finally, taking (52) and (49) into (46) to give

$$\frac{1}{|\Omega_0|}\left(\int_{S_H} -\frac{1}{|\nabla_0 H_0|}\frac{\partial H_0}{\partial t_0}dS_0 + \overline{K}_j\frac{\partial p_0}{\partial x_{1j}}\right) = 0 \quad (53)$$

So far, we have obtained the averaged 0th-order continuity. Before continuing the derivation, we need to define water content first. As defined in (14), the water content is

$$\theta_0(x_{1i}, x_{2i}, \ldots, t) = \frac{\Omega_{l0}}{|\Omega_0|} = \frac{1}{|\Omega_0|}\int_0^{H_0} A dz_0 \quad (54)$$

where A = A(z) is the area of water with z variation in a REV. Water content θ_0 depends on $x1i$, $x2i$,... , t , and it can be regarded as the averaged water content in macro-scale; and H0 is function of time in unsaturated soil. Differentiating (54) with respect to t0 once, we could obtain

$$\frac{\partial \theta_0}{\partial t_0} = \frac{1}{|\Omega_0|}\frac{\partial \Omega_{l0}}{\partial t_0} = \frac{A(H_0)}{|\Omega_0|}\frac{\partial H_0}{\partial t_0} \quad (55)$$

where A (H_0) is the area at free surface of water in REV. Using (55) and substituting it into (53), we obtain

$$\frac{\partial \theta_0}{\partial t_0}\int_{S_H}\frac{dS_0}{|\nabla_0 H_0| A(H_0)} = \frac{K_j}{|\Omega_0|}\frac{\partial p_0}{\partial x_{1j}} \quad (56)$$

The inner product of unit normal vector on the surface of infinitesimal area, dS_0 , together with the unit z-direction vector, $e_z = (0,0,1)$, is

$$\cos\theta = \frac{\nabla_0 \mathrm{H}_0 \cdot e_z}{\left|\nabla_0 \mathrm{H}_0\right|\left|e_z\right|} = \frac{1}{\left|\nabla_0 \mathrm{H}_0\right|} \tag{57}$$

where θ is the angle between unit normal vector of free surface and z-direction vector. (see Fig. 2)

Therefore, the integrand inside the integral of LHS in (56) becomes

$$\frac{dS_0}{\left|\nabla_0 \mathrm{H}_0\right|} = dS_0 \cos\theta \tag{58}$$

In (58), $dS_0 \cos\theta$ is the projection area of free surface of water on the z-plane. By using (58), (56) becomes

$$\frac{\partial \theta_0}{\partial t_0} \int_{s_H} \frac{\cos\theta dS_0}{\mathrm{A}(\mathrm{H}_0)} = \frac{\overline{K}_j}{\left|\Omega_0\right|} \frac{\partial p_0}{\partial x_{1j}} \tag{59}$$

(59) is another form of averaged continuity of 0^{th}-order REV. In the micro-scopic REV, the projection area $dS_0 \cos\theta$ approaches the original area, $\mathrm{A}(\mathrm{H}_0)$ in the macroscopic point of view. Therefore, integral on LHS is very close to 1, and finally (59) becomes

$$\frac{\partial \theta_0}{\partial t_0} = \mathrm{K}_x \frac{\partial p_0}{\partial x_1} + \mathrm{K}_y \frac{\partial p_0}{\partial y_1} + \mathrm{K}_z \frac{\partial p_0}{\partial z_1}$$
$$\text{where } \mathrm{K}_i = \frac{\overline{K}_i}{\left|\Omega_0\right|} = \frac{1}{\left|\Omega_0\right|} \int_{s_j} K_{ji} dS_0 \tag{60}$$

K_i could be regard as the REV-averaged hydraulic conductivity in x, y and z-direction respectively. And from (60), we could find that the time rate of change of water content in the macro-scale is proportional to the macroscopic pressure gradient in each direction. If we have the macroscopic boundary conditions of pressure together with the hydraulic conductivity, from the experiments or other calibrated data, in each direction of soil, we could use (60) to obtain the water content in soil. (60) is the same as the Richards' equation(1931).

CONCLUSION AND DISCUSSION

In the problem of the seepage flow in unsaturated static soil, without any constitutive assumption for solid-liquid mixture, we successfully use homogenization theory to obtain the equation governs water content which is proved to be the same as Richards' equation. From this result, we could conclude that homogenization theory is adequate to be used in the problem of unsteady and unsaturated solid-liquid mixtures. However, the result in this paper is only the leading order solution for fixed solid. We will continue using this theory in the problem of unsteady, unsaturated and movable solid of solid-liquid mixture to derive the macro-scopic motion of solid-liquid mixture and study the process of debris-flow initiation.

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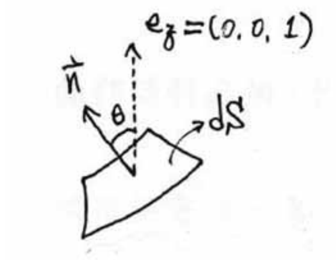

Fig. 2 - Unit normal of infinitesimal surface and e_z

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