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STEADY DEBRIS FLOWS OVER ERODIBLE BEDS

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ABSTRACT

Recently, Berzi & Jenkins (2008a, b) proposed a simple theory based on a linear rheology for the particle interactions, turbulent shearing of the fluid, buoyancy, and drag. They provided a complete analytical description of the steady, uniform flow of a granular-fluid mixture over either an erodible or a rigid bed contained between frictional sidewalls. They also used the theory to solve for the propagation of a granular-fluid wave moving at constant velocity over a rigid bed.

Here, we extend this theory to the case of a granular-fluid wave moving at constant velocity over an erodible bed contained between frictional sidewalls. This is indeed a natural step in view of a realistic mathematical description of a real debris flow that propagates over mobile surfaces, where erosion/deposition phenomena are likely to occur. We make comparisons with the experiments performed with water and gravel and show that the theory is able to reproduce the wave front and body.

Key words: steady wave, erodible bed, rheology

INTRODUCTION

Natural debris flows typically consist of unsteady, non-uniform surges of heterogeneous mixtures of muddy water and high concentrations of rock fragments of different shapes and sizes, driven down a slope by gravity. Despite that, our intent here is to emphasize steady flows of idealized composition.

Berzi & Jenkins (2008a, b) developed a simple theory based on a linear rheology for the particle interactions, turbulent shearing of the fluid, buoyancy, and drag. They assumed a constant concentration in the particle-fluid mixture and the similarity of the particle and fluid velocity profiles to obtain a complete analytical description of the steady, uniform flow of a granular-fluid mixture over an inclined bed contained between frictional sidewalls. The predictions of this description compared favourably with the measurements in experiments on steady, uniform granular-fluid flows performed by Armanini *et alii* (2005) and Larcher *et alii* (2007) on mono-dispersed plastic cylinders and water. Berzi & Jenkins (2009) used their theory to solve for the propagation of a steady granular-fluid wave over a rigid bed and were able to reproduce the experiments performed by Davies (1988) on mono-dispersed plastic cylinders and water. The theory was further simplified to obtain explicit expressions for the particle and fluid friction slopes as functions of the particle and fluid depth-averaged velocities and depths to be employed in mathematical models (Berzi *et alii*, 2010).

Here, we extend the theory of Berzi & Jenkins (2008a, b; 2009) to deal with the steady propagation of a granular-fluid wave over a previously deposited erodible bed. We assume that the ratio of the particle shear to normal stress is distributed as in uniform flows. This indeed allows to determine the position of the interface between the flowing layer and the erodible bed, where we assume that the granular material is at yield. Using the set of model parameters appropriated for the uniform flow of 3 mm gravel and water, as suggested by Berzi *et alii* (2010), we show that the theory is able to reproduce the experimental wave profile measured by Tubino & Lanzoni (1993).

The paper is organized as follows: first, we present the depth-averaged equations governing the motion of a steady granular-fluid wave over an erodible bed and the closures for the particle and fluid resistances and the location of the bed on the basis of the theory of Berzi & Jenkins (2008a, b; 2009) and Berzi *et alii* (2010); then we show the comparisons of the theory against experiments on steady uniform and non-uniform flows of gravel and water over erodible beds performed by Tubino & Lanzoni (1993). Finally, we point out some concluding remarks.

GOVERNING EQUATIONS AND CLOSURES

Figure 1 shows the sketch of the flow configuration. We let ρ denote the fluid mass density, g the gravitational acceleration, σ the particle specific mass, d the particle diameter and η the fluid viscosity. The Reynolds number R = $\rho d_{,}(gd)^{1/2}/\eta$ is defined in terms of these. In what follows, we phrase the momentum balances and constitutive relations in terms of dimensionless variables, with lengths made dimensionless by d, velocities by $(gd)^{1/2}$, and stresses by $\rho\sigma gd$.

We take x and z to be the coordinates parallel and perpendicular, respectively, to the initially plane erodible bed of inclination ϕ with respect to the horizontal; $z = b$ is the position of the erodible bed (with $b = 0$ downstream of the wave), while $z = h$ and $z = H$ are the top of the particles and the fluid, respectively. The degree of saturation, $\zeta \equiv H/h$, is greater than unity in the over-saturated flows and less than unity in the under-saturated. U_A is the depth-averaged fluid velocity, and u_A the depth-averaged particle velocity. The particle and fluid snouts are at $x = x^*$ and $x = X^*$, respectively. We consider here a steady granular-fluid wave, so that h, H and b are functions of position x, but not of time t. As in berzi & jenkins (2009), we assume that the flow is dense and at approximately constant concentration, $\hat{c}$; we also assume the yielding of the particles at the interface with the erodible bed and that the ratio of the particle shear to normal stress there is equal to $\breve{u}$.

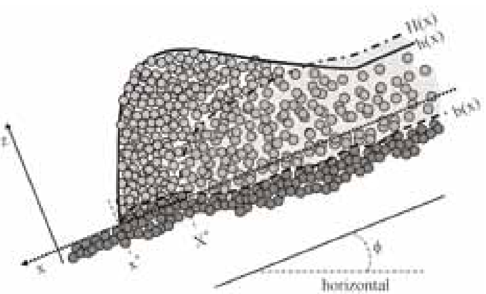

Fig. 1 - Sketch of the flow configuration with the frame of reference

As depicted in Fig. 1, we can distinguish a wave front, where the flow is strongly non-uniform, and a body, where $h(x)$, $H(x)$ and $b(x)$ are approximately rectilinear and parallel, as in uniform flows; this configuration has been experimentally observed by Davies (1988) and Tubino & Lanzoni (1993). We make use of the so called uniformly progressive wave approximation (Hungr, 2000; Pouliquen, 1999) to obtain the shape of the front; i.e., we assume that the depth-averaged particle and fluid velocities are equal and constant, as in Berzi & Jenkins (2009) and Berzi *et alii* (2010).

The depth-averaged momentum balances for the particles and the fluid are

$$\frac{\mathrm{d}(h-b)}{\mathrm{d}x} + \frac{\alpha}{\sigma}\frac{\mathrm{d}\left[(\beta-1)(h-b)\right]}{\mathrm{d}x} = \tan\phi - \frac{\mathrm{d}b}{\mathrm{d}x} - j\,, \qquad (1)$$

and

$$\frac{\mathrm{d}(H-b)}{\mathrm{d}x} = \tan\phi - \frac{\mathrm{d}b}{\mathrm{d}x} - J\,, \qquad (2)$$

respectively, where α and β are functions of the degree of saturation, so that, when the flow is under-saturated, $\alpha = H/h$ and $\beta = 1$, and, when the flow is over-saturated, $\alpha = 1$ and $\beta = H/h$. With respect to the corresponding equations governing the motion of a steady granular-fluid wave over a rigid bed (Berzi & Jenkins, 2009), we use the local inclination of the erodible bed, tan [ϕ - arctan (db / dx)] ≈ tan ϕ - db / dx, valid if db / dx is small, to account for the component of the weight in the direction of the flow.

In Eqs. (1) and (2), j and J are the particle and fluid friction slopes, respectively; they summarize the resistances due to internal shear stresses and the role of the drag force. Berzi *et alii* (2010) express them as

$$j = \frac{1}{\lambda_1}\frac{u_A}{\tilde{h}^{3/2}} + \frac{\lambda_2}{\lambda_1}\,, \qquad (3)$$

and

$$J = \left\{\frac{-\Lambda_3 + \left[\Lambda_3^{\,2} + 4\Lambda_1\left(\Lambda_2\tilde{H}^2 + U_A\tilde{H}^{1/2}\right)\right]^{1/2}}{2\Lambda_1\tilde{H}}\right\}^2, \qquad (4)$$

where $\tilde{\mathrm{h}} \equiv h - b$ and $\mathrm{H} \equiv \tilde{H} - b$. The coefficients λ_1, λ_2,

λ_1 if $j \le \breve{\mu}$	$\frac{2}{15\chi\bar{c}\sigma^{1/2}(\sigma-1)^3}\left\{\left[(3\sigma-5+2\alpha)(\sigma-\alpha)^{3/2}-\sigma^{3/2}(3\sigma-5)(1-\alpha)^{5/2}\right](\bar{c}\sigma+1-\bar{c})\right.$ $\left.+5\left[(\sigma-3+2\alpha)(\sigma-\alpha)^{1/2}-\sigma^{1/2}(\sigma-3)(1-\alpha)^{3/2}\right]\left[(\sigma-1)(\beta-1)-\sigma(1-\alpha)\right]\right\}$
λ_1 if $j > \breve{\mu}$	$\frac{2}{15\chi\bar{c}\sigma^{1/2}(\sigma-1)^3}\left\{\left[(3\sigma-5+2\alpha)(\sigma-\alpha)^{3/2}-\sigma^{3/2}(3\sigma-5)(1-\alpha)^{5/2}\right](\bar{c}\sigma+1-\bar{c})\right.$ $+5\left[(\sigma-3+2\alpha)(\sigma-\alpha)^{1/2}-\sigma^{1/2}(\sigma-3)(1-\alpha)^{3/2}\right]\left[(\sigma-1)(\beta-1)-\sigma(1-\alpha)\right]$ $\left.+3\bar{c}\sigma^{1/2}(\sigma-1)^3(1-\alpha)^{5/2}\right\}$
λ_2 if $j \le \breve{\mu}$	$\frac{2}{15\chi\sigma^{1/2}(\sigma-1)^2}\left[(3\sigma-5+2\alpha)(\sigma-\alpha)^{3/2}-\sigma^{3/2}(3\sigma-5)(1-\alpha)^{5/2}\right]\breve{\mu}$
λ_2 if $j > \breve{\mu}$	$\frac{2}{15\chi\sigma^{1/2}(\sigma-1)^2}\left[(3\sigma-5+2\alpha)(\sigma-\alpha)^{3/2}-\sigma^{3/2}(3\sigma-5)(1-\alpha)^{5/2}\right.$ $\left.+3\sigma^{1/2}(\sigma-1)^2(1-\alpha)^{5/2}\right]\breve{\mu}$

Tab. 1 - Coefficients in Eq. (3).

Λ_1	$\frac{2(1-\bar{c})}{15\xi^{3/2}\chi\bar{c}\sigma^{1/2}(\sigma-1)^3\left[\alpha(1-\bar{c})+\beta-1\right]}\left\{\left[(5\alpha\sigma-3\alpha-2\sigma)(\sigma-\alpha)^{3/2}+2\sigma^{5/2}(1-\alpha)^{5/2}\right.\right.$ $\left.+5(\sigma-1)^{5/2}(\beta-1)\right](\bar{c}\sigma+1-\bar{c})$ $+5\left[(3\alpha\sigma-\alpha-2\sigma)(\sigma-\alpha)^{1/2}+2\sigma^{3/2}(1-\alpha)^{3/2}\right]\left[(\sigma-1)(\beta-1)-\sigma(1-\alpha)\right]$ $\left.+15(\beta-1)^2(\sigma-1)^{5/2}\right\}$
Λ_2	$\frac{2(1-\bar{c})}{15\xi^{3/2}\chi\sigma^{1/2}(\sigma-1)^2\left[\alpha(1-\bar{c})+\beta-1\right]}$ $\times\left[(5\alpha\sigma-3\alpha-2\sigma)(\sigma-\alpha)^{3/2}+2\sigma^{5/2}(1-\alpha)^{5/2}+5(\sigma-1)^{5/2}(\beta-1)\right]\breve{\mu}$
Λ_3	$\frac{2(\beta-1)^{3/2}}{5\xi^{1/2}k\left[\alpha(1-\bar{c})+\beta-1\right]}$

Tab. 2 - Coefficients in Eq. (4).

Λ_1, Λ_2 and Λ_3 are reported in Tables 1 and 2 (from Berzi *et alii*, 2010). There, χ is a material coefficient of order unity that characterizes the linear rheology for the particles adopted by Berzi & Jenkins (2008a, b; 2009), while k = 0.2 (half the Karman's constant). Equations (3) and (4) have been obtained in uniform flow conditions, but, as usual in Hydraulics, we use them also in the case of non-uniform motion.

To close the problem, we need an equation governing the evolution of the position of the erodible bed. Berzi & Jenkins (2008b; 2009) derive the distribution of the ratio of the particle shear to normal stress, along the cross-section of the flow, in the case of steady, uniform motion over erodible beds. They link the inclination of the erodible bed to the particle and fluid heights above it and emphasize the role of frictional sidewalls, characterized by their friction coefficient μ_w and gap W, in locally controlling the particle stress ratio. If we assume that. at the erodible bed, the stress ratio is equal to the yielding value, ŭ, and we use the local inclination of the bed, , the expression of Berzi & Jenkins (2008b; 2009) reads

$$\tan\phi-\frac{\mathrm{d}b}{\mathrm{d}x}=\frac{\bar{c}(\sigma-\alpha)}{\bar{c}\sigma+\alpha(1-\bar{c})+\beta-1}\breve{\mu}+\frac{\bar{c}(\sigma-\alpha^2)}{\bar{c}\sigma+\alpha(1-\bar{c})+\beta-1}\frac{\mu_w}{W}(h-b) \tag{5}$$

Equations (1), (2) and (5) are three ordinary differential equations governing the spatial evolution of h, H and b. Their integration requires the knowledge of three boundary conditions; a natural choice would be the vanishing of the particle and fluid heights at the snouts, $h(x^*) = H(X^*) = 0$, and the fact that the bed is unperturbed downstream, $b(\max[x^*, X^*]) = 0$.

COMPARISONS WITH EXPERIMENTS

We now make comparisons between the present theoretical treatment and the experiments performed by Tubino & Lanzoni (1993) with water and gravel, of specific mass σ = 2.65 and diameter d = 3 mm, flowing over erodible beds in a rectangular channel of width W = 67 diameters, and contained between glass sidewalls. Here, we use ŭ = 0.52 and ĉ = 0.60, as suggested by Berzi *et alii* (2010), and χ = 1 and μ_w = 0.3. The latter two values are slightly different than those adopted by Berzi *et alii* (2010), but they allow for a better fitting of the experiments in uniform flow conditions.

We first show the capability of the theory to reproduce the experimental results in uniform flow conditions. If we take the derivative with respect to x to be zero in Eqs. (1), (2) and (5), we analytically obtain h, H and u_A (or equivalently, the particle flow rate per unit width, $q = \hat{c}u_A h$) as functions of tanϕ and U_A (or equivalently the fluid flow rate per unit width, $Q = [(1 - \hat{c})\,\alpha + \beta - 1]\,U_A h$). Figures 2 and 3 show the comparisons between the theory and the experimental measurements of q and h as functions of tanϕ. Given that the experiments are for a range of fluid flow rate of 11.7 to 27.2, we use the average value, Q = 19.5, to obtain the analytical results. The agreement is notable and suggests that the theory can be used to predict the characteristics of the body of the debris flow depicted in Fig. 1, where the motion is approximately uniform.

We then numerically solve the full differential equations (1), (2) and (3) using a fourth-order Runge-Kutta method to see if the theory has the capability to reproduce also the wave front. Tubino & Lanzoni (1993) reported measurements of the wave height as a function of time t for one of their experiments. For that experiment, where the inclination of the undisturbed erodible bed was 17°, they also measured the front velocity and found it constant and equal to 0.476 m/s, corresponding to a nondimensional velocity of 2.8. If the flow is steady, then x = 2.8t, and we can compare the experimental measurements with the results of the present theory.

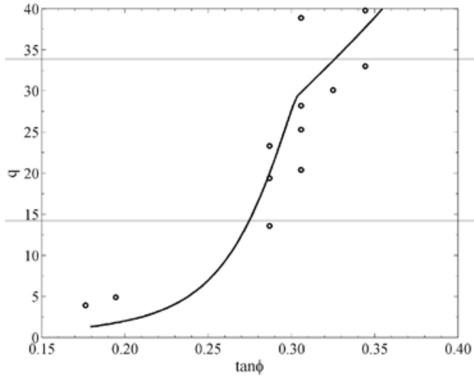

Fig. 2 - Theoretical (solid line) and experimental (circles, from Tubino & Lanzoni , 1993) particle flow rate against the angle of inclination of the erodible bed. The theoretical results are for Q = 19.5

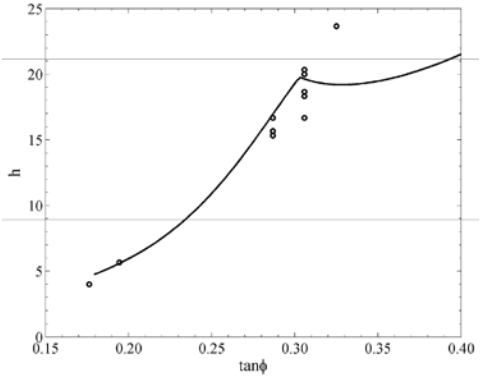

Fig. 3 - Same as in Fig. 2, but for the particle depth against the angle of inclination of the erodible bed

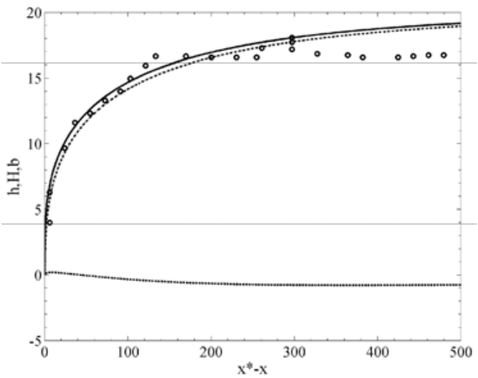

Fig. 4 - Theoretical prediction of the spatial evolution of the top of the particles (solid line), the top of the fluid (dot-dashed line) and the position of the erodible bed (dashed line) against the experimental measurements (circles, from Tubino & Lanzoni, 1993) of the profile of a steady wave over an erodible bed, for $u_A = U_A = 2.8$ and $\phi = 17°$

The erodible bed in the experiments of Tubino & Lanzoni (1993) was initially saturated with water; moreover, they describe the debris flow as being fully saturated, i.e. with the height of the particles over the bed approximately equal to the height of the fluid.

We therefore assume here that the particle and fluid snouts coincide, $x^* = X^*$, and we solve Eqs. (1), (2) and (3) with $u_A = U_A$ =2.8, to obtain, at every step of integration, the values of the friction slopes from Eqs. (3) and (4), and ϕ = 17°. In Fig. 4, we show the predictions of the theory against the experiment of TTubino & Lanzoni (1993). The free surface of the wave is well reproduced by our numerical solution. Also, the position b of the interface with the erodible bed is positive in the wave front and is negative upstream; this indicates that the debris flow tends to deposit material at the front and to erode it at its upstream end, in accordance with the experimental observations of Tubino & Lanzoni (1993).

CONCLUDING REMARKS

We have extended the two-phase theory of Berzi & Jenkins (2008a, b; 2009) to deal with steady, non-uniform debris flows over erodible beds contained between frictional sidewalls. This flow configuration represents a severe test to the practical applications of the theory to real scale phenomena. The theory can predict, beside the heights of the particles and the fluid, also the spatial evolution of the position of the interface between the flow and the erodible bed.

The comparisons of the theoretical results with the experiments performed using natural gravel and water is remarkably good. The theory is able to quantitatively predict both the front and the body of the steady waves; it also confirms the experimentally observed tendency of debris flows to be depositional at the front, and erosional upstream.

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