The Relationship of Savings to Some Important Variables in the Business Cycle

by

MONTGOMERY D. ANDERSON

I

The cash savings of an individual are by common-sense definition equal to the difference between his cash income and his cash expenditures. Thus if a salaried man with no property income received \$ 500 cash for his services in a given month and expended a total of \$ 400, he would consider that he had saved \$ 100 in cash that month. His *rate* of cash saving at any given moment, t may be designated by $s(t)$, while his rate of cash income and rate of cash expenditures at the same moment may be represented by $i(t)$ and $x(t)$ respectively (1). All three of these variables will be of the dimension dollars (or lire, etc.) per period of time. Then, from the common-sense point of view, one may write

$$s(t) = i(t) - x(t), \qquad [1]$$

which states algebraically that the rate of cash saving at a given moment is equal to the difference between rate of cash income and rate of cash expenditure.

Now, if any given individual, natural, corporate, or governmental, does not bring precious metal to the mint, does not print paper money, and does not borrow or lend (or steal), then his cash savings will be equal to the increase in his stock of money. If $m'(t)$ represents the rate of increase of his stock of money then under the stipulated restrictions one may write the following equation:

$$m'(t) = s(t) = i(t) - x(t). \qquad [2]$$

Let $m(t)$ be a continuous function of time with continuous derivatives which approximates the amount of money an individual has at any given moment. The existence of this function is, of course, implied by the use of $m'(t)$ in formula [2], because $m'(t)$ in the instantaneous rate of change of m. Let α be a variable function of time which is *defined* by the following definite integral:

$$m(t) = \int_{t-\alpha}^{t} i(t)\,dt, \qquad [3]$$

while restrictions above are maintained.

Formula [3] requires the cash income (*i.e.*, the rate of receipt of cash, under the restrictions) to be integrated backward in time from any given moment, t until such time is reached that the accumulation of past receipts of cash is exactly equal to the stock of money held by the individual at time t. This lower limit

(1) In strict logic most individuals do not have a continuous rate of cash income, as is implied by writing i as a function of time t, because they receive their cash in « lumps » at intervals of time, with no receipts at all during the interval. However, one may conceive an individual to have a *constructive* rate of cash income which is a continuous function of time in this fashion: accumulate all the cash income of the individual from birth to any given time, listing a value for each date on which money was received. Now fit a smooth curve to this accumulated series and differentiate the curve. The resulting derivative will be a continuous function of time, and its value at any given moment may be said to represent the constructive rate of cash income at that moment for the given individual, in the sense that the definite integral of this function over any considerable duration of time will give a close approximation to the amount of cash received by the individual during the period of time covered by the integration.

From similar considerations one may conceive an individual to have an instantaneous rate of expenditure of cash. The difference between these two rates will give the rate of cash saving for the individual in the sense that the definite integral of the difference between the two functions of time, when taken over any considerable period of time, will approximate the difference between cash income and cash expended.

of the integral is designated by $t-\alpha$, and α is thus defined as the interval of time over which the integration must be carried out in order to reach the result demanded by formula [3].

A little thought will show that α is of the nature of the duration of time that the individual holds his money before spending it. Since money (cash) is essentially homogeneous, one may think of it as passing through the hands of an individual according to the rule of « first in, first out ». Then, so long as the rate of cash expenditure has any positive value, the interval α can be thought of as the length of time the dollar just about to be expended at time t has remained in the possession of the given individual (2). Under the rule stated, whatever cash was on hand at time $t-\alpha$ has been « pushed out » of pocket (or bank account) by the money that was acquired between time t and time $t-\alpha$. The dollar just about to be spent at time t was acquired at time $t-\alpha$.

Whether or not money really does follow the rule of « first in, first out », *i.e.*, whether people do in point of fact put incoming currency in the bottom of their wallets, and spend from the top is wholly immaterial to the argument of this paper. In any case the interval α has the conceptual properties of a *period of turnover* of money, and the hypothetical rule of inventorying was invoked above merely as an aid to understanding by the reader.

The rate of change of an individual's stock of money, $m'(t)$, is given by taking the derivative of m with respect to time, *i.e.*, by differentiating formula [3] above. This gives the following result:

$$m'(t) = [i(t) - i(t-\alpha)] + [\alpha' i(t-\alpha)] = s(t), \qquad [4]$$

where α' is the rate of change of α.

The assumed prohibitions against borrowing, printing money, etc., are still in effect in the argument, and therefore the rate of increase of the individual's stock of money is equal to his rate of cash saving $s(t)$.

Formula [4] expresses the value of cash savings in terms of cash income, the period of turnover of the individual's money, and the rate of change of that period of turnover. It is explicitly independent of the rate of cash expenditure because the effect of this variable has been implicitly subsumed by the behavior of α and α'.

Be it noted that the common-sense definition of cash savings has been transformed by equation [4] into two very different economic activities, the dollar values of which are given by the terms contained in the pair of brackets on the right-hand side of equation [4].

This equation shows there are two very different ways in which an individual can save money. On the one hand he can bestir himself and increase his cash income. The rate of cash saving produced by this form of action is given by the first bracket in formula [4] because it evaluates the difference between rate of cash income at the given time t and the rate of cash income at an earlier time, $t-\alpha$. The other way an individual can save money is to be more sparing in the use of the money which he has; in other words hold his money a longer time before spending it. The rate of cash saving produced by this procedure is evaluated by the second bracket in formula [4] because it has a value equal to the product of the rate of change of the period of turnover of the person's money multiplied by the rate of cash income at time $t-\alpha$. Increasing the period of turnover of cash will increase cash savings, ceteris paribus, because there is a plus sign in front of the second bracket, and the sign of α' will be positive when α is increasing.

The restrictions against borrowing, etc., are still maintained in the argument. In that case it has been demonstrated mathematically that the cash savings of an individual are *always* a function of the period of turnover of his money because this variable appears in *both* the bracket terms of formula [4]. Should an individual's cash income be constant then his rate of cash saving becomes *entirely* a function of the period of turnover of his money and the rate of change of that period because the first bracket term in formula [4] vanishes, and the following special case obtains:

$$m''(t) = s(t) = \alpha' \beta \qquad [5]$$

where β equals the constant cash income. With cash income constant variation in cash savings is wholly determined by variation in the period of turnover of money. Let u be the *rate* of turnover (or velocity) of an individual's money. There are several ways in which u might be evaluated. One logical way is to define u as the *reciprocal* of the period of turnover of money, α. In that case formula [5] becomes

$$m'(t) = s(t) = \frac{-\beta u'}{u^2} \qquad [6]$$

where u' is the rate of change of the rate of turnover of an individual's money. Thus in the case where cash income is constant the cash savings of an individual are an inverse function of the rate of change of the velocity of his money. Since u itself is always positive the sign of the cash savings will be the reverse of the sign of u'. If an individual increases the rate of turnover of this money while his cash income is constant his cash savings will be negative. This, of course, is just common sense.

II

In order to evaluate the total value of the cash savings of all persons, natural, corporate, and governmental, let upper case letters be employed to match the lower case letters used for individuals. Then by analogy with formula [1] the rate of cash savings of all persons would be written as

$$S(t) = I(t) - X(t). \qquad [7]$$

These aggregates are, of course, continuous functions of time and it is not necessary to employ a set of constructs in order to give them continuity as was necessary in the case of an individual. The cash savings of every person in the aggregate will, of course, always be zero by virtue of the fact that $I(t)$ is identically equal to $X(t)$. Every dollar of cash income received by one person must have been expended by some other person: hence $I(t) = X(t)$.

In a society of traders where no new metal is coined, no paper money is printed, and no derivative bank deposits are created the rate of change of total stock of money will also be zero, neglecting currency that is lost or melted.

But in a society wherein these restrictions are removed the rate of change of total stock of money may show a positive value while the cash savings are still zero. That is because without the restrictions cash *receipts* may be larger than cash income. Let $Z(t)$ represent the aggregate rate of cash receipts, and let M equal the total stock of money (cash). Then by analogy with formula [3] one may write the following equation:

$$M(t) = \int_{t-A}^{t} Z(t)\, dt. \qquad [8]$$

From considerations similar to those used to explain the meaning of α in formula [3] it may be seen that A is the average period of turnover all money including money received from sources other than income. The rate of change of total stock of money, M' is given by differentiating formula [8] with respect to time, viz.,

$$M'(t) = [Z(t) - Z(t-A)] + [A' Z(t-A)]. \qquad [9]$$

where A' is the rate of change of the average period of turnover of all money.

Now put the familiar restrictions back into the argument and revert to an imaginary society in which there is no change in the quantity of money by virtue of the assumed restrictions. In that case $Z(t) = I(t)$; *i.e.*, aggregate cash income equals cash expenditures. Hence, the aggregate rate of increase of money $M'(t)$ equals the total rate cash saving $S(t)$ because both these aggregates are equal to zero. Hence in this imaginary country one may substitute $S(t)$ for $M'(t)$ in equation [9], viz.,

$$S(t) = [Z(t) - Z(t-A)] + [A'Z(t-A)] = O. \qquad [10]$$

Since Z, the rate of receipt of cash for all traders, is always greater than zero it follows that the algebraical sign of the net value of

(2) Since the accumulated expenditures data will be increasing monotonically there will be no points in time at which the derivative of the fitted curve of expenditures will not be positive. In other words the individual will always be spending at least a little money according to the constructed function $x(t)$ even though in reality he may go several days without any expenditure.

the terms in the first bracket is always opposite to the sign of the net value of the second bracket in formula [10], that is to say so long as one maintains the restrictions against the creation of money. Algebraically stated [10] signifies that

$$Z(t) - Z(t - A) = - A' Z(t - A). \qquad [11]$$

Thus for traders as a whole the aggregate cash savings achieved by one type of economic operation are always exactly defeated by and canceled out by another type of operation, either one of which would increase total cash savings, *if* it could be done without the other. But one type of operation implies the existence of the other economic effect *somewhere* in the economy. If a given trader saves cash by holding his money a longer time before spending it — thereby causing his individual contribution to the second bracket in formula [10] to be positive — he will automatically cut down on the cash income of some of the people from whom he buys. This will make a negative contribution to the aggregates contained in the first bracket term. Thus the positive saving of the first mentioned person will be exactly canceled by negative saving on the part of the last named group of persons *unless* they in turn decrease the rate of turnover of *their* money and thereby cause a negative saving for still other persons. Somewhere down the line the cycle must balance out, but it may be a long way down the line.

Thus one comes to understand why a vicious circle or chain reaction tendes to perpetuate a business depression onbe it gets started. Each trader tries to protect his cash balance — at some point of resistance — from the inroads made upon it by the cash savings of other traders. Thus is instanced one of the great paradoxes of a money economy: that a perfectly legal operation which enhances a given individual's wealth may detract from that of his neighbor's.

The most important conclusion of the foregoing analysis is that the common-sense definition of cash savings turns out to be not very sensible when applied in the aggregate because (a) it is identically equal to zero in the aggregate, and (b) because it is composed of two quite heterogeneous and antithetical operations even for the individual case.

As an alternative to the common-sense definition (cash income minus cash expenditures) one might define the cash savings of all traders as a value equal to the rate of increase of their cash holdings, M'. Aggregate cash savings would then be equal to new metals minted plus new currency printed plus the net change in derivative bank deposits minus coin and currency lost. In a modern society such definition would make the value of cash savings for all traders depend almost entirely on treasury and bank policy. Therefore, this alternative definition does not seem to be very sensible or very useful.

The most sensible and useful way to define aggregate rate of cash savings would be to identify it not with the *whole* rate of increase of stock of money M', but rather with *that part* of the rate of change of stock of money to which a positive value is contributed by the second bracket in formula [9]. This bracket evaluates the increase in cash holdings which is brought about by economizing in the use of cash, *i.e.*, by lengthening the average period of turnover of money. The *other* part of the whole rate of increase of aggregate stock of money, which is evaluated by the first bracket in formula [9], could be called the rate of *new cash income*. Under the definitions just stated total cash savings would measure the aggregate effect of homogeneous individual operations rather than the aggregate effect of heterogeneous and conflicting operations, as in the case of the common-sense definition.

Moreover, there could be either positive or negative cash savings in the aggregate without any change in the total stock of money, *i.e.*, in the case where $M' = O$. To say that a *part* of nothing could be positive is, of course, perfectly good mathematical logic. The other part — the new cash income — would be negative and of the same numerical value, in that circumstance. But when there is an increase of total stock of money cash savings and new income *could* both be positive as far as the mathematics are concerned.

However, even when the restrictions against printing paper money, etc., are relaxed, so that M' is not necessarily zero it still remains true that the two bracket terms in formula [9] represent opposing economic tendencies to a large extent. In practice these two bracket terms tend to have opposite signs in the aggregate. Thus when new cash income is positive and business men are optimistic because of this fact the cash savings bracket will usually be negative. Under the optimistic impulse the propensity to spend increases, the period of turnover of cash decreases and cash savings as defined by the second bracket term in equation [9] will usually be negative.

A definition of cash savings for the individual would be obtained by substituting lowercase letters for the capital letters in formula [9]. The ordinary person could never be persuaded to apply this formula to compute his cash savings. But neither can the ordinary individual analyse any scientific phenomenon accurately and with the most functional definitions. The scientist — the economist — need not be deterred by the fact that his mode of thinking is not that of the layman, although to be sure the economist does have to make himself understood by the layman more often than does the physicist, for example. The ordinary person will continue to calculate his cash savings by taking the difference between cash income and cash expenditures.

But in case the said ordinary individual should happen to have a fixed cash income, and neither borrow money nor steal it, then ***he will unwittingly compute his cash savings as defined by the second bracket of formula*** [9], because in that case the common-sense formula will by identical with the second bracket term in formula [9].

However, for the economist to apply the proposed definition in appraisal of the cash savings of an individual or corporation he would in general need to use more elaborate statistics than required by the common-sense definition (3). This is admittedly a handicap, but it should not be controlling because the proposed new definition has one overwhelming advantage, to-wit, that it is not identically equal to zero in the aggregate and therefore is not completely meaningless for society as a whole, as is the conventional definition.

The proposed new definition of cash savings has a secondary advantage that is nearly as great as the first one; namely, that it does not lump together two economic operations which are essentially different for the individual and in the aggregate.

The mathematical antithesis existing between new cash income and cash savings as defined in this paper may be revealed very neatly by factoring the Z term in formula [9] into the product of M, the total stock of money, and V its average velocity, where V is defined as Z/M. In that case formula [9] becomes

$$M'(t) = \left[\int_{t-A}^{t} M'(t)V(t)dt + \int_{t-A}^{t} V'(t)M(t)dt\right] + [A' Z(t - A)]. \qquad [12]$$

where V' is the rate of change of V with respect to time.

The average rate of turnover of money as measured by V will not in general be exactly the same as the average rate of turnover obtained by taking the reciprocal of A, the average period of turnover of money. V and $1/A$ are in general slightly different concepts of average velocity. However V will always vary inversely with respect to A; so the algebraical sign of V' will always be opposite to that of A'. The second integral in the first bracket of [12] and the second bracket each evaluate numerically the contribution of changes in rate of turnover of money to increase of cash receipts while total money remains constant. The former evaluates this contribution positively and the latter reversely. Hence the second integral in the first bracket always cancels out exactly the second bracket. This shows the antithetical nature of the first and second brackets. While not completely antithetical they are inevitably opposed to a certain extent because the first contains a term which is identically equal to the second bracket, but of opposite sign, unless they are both equal to zero.

The only thing which prevents new cash income from being completely antithetical to

(3) A fairly quick way to compute the approximate cash savings of an individual or firm whose income is variable would be as follows:

1) Compute the increase of cash during the period.

2) Estimate the period of turnover of the firm's money at the moment ending the perod.

3) Using this estimate compute the new cash income for the period as evaluated by the first bracket in formula [9].

4) Subtract 3) from 1).

cash savings as defined in this paper is the presence of the first definite integral in the first bracket. This integral evaluates the rate of receipt of new money and must always be of equal value and of the same sign as the rate of change of stock of money, M'. In other words the rate of increase of money is the same thing as the rate of receipt of new money when « new » money is considered to be the gain in stock of money over the interval A. The first integral in formula [12] indicates that the increase in money is turning over because it imparts a constructive rate of turnover to the term M' equal to the rate of turnover of old money and then quietly cancels out this multiplying effect of the V by integrating the product over the period A, leaving just M' after all (4).

So one can see that when cash savings are positive new cash income will *have* to be negative *unless* the stock of money is increasing, and increasing sufficiently for the first integral in the first bracket of formula [12] to overcome the negative effect of the second integral.

This last is true for the individual as well as for society as a whole. But it does not mean the individual cannot increase his stock of cash by saving money in the way it is defined by this paper. If he saves cash by this definition, he will slow down the rate of turnover of his money. If at the same time he manages to hold his cash receipts constant, all *three* of the major groupings in formula [12] will be numerically equal, and his increase of cash will be due entirely to the savings operation, provided he does not borrow or print money, etc. Although the savings operation of the individual will be canceled exactly by the effect measured by the second definite integral in formula [12] the operation will give rise to a positive value in the first definite integral which is uncanceled and will be numerically equal to the increase of cash. In this special case M', the first and second integrals and the second bracket all have the same numerical value, and cash savings equal increase of cash for the individual, subject to the familiar restrictions.

(4) Where M' is constant it is easy to show that the value of the first integral in formula [9] equals M'. In that case the value of the first integral becomes equal to

$$M'V(\Theta) \int_{t-A}^{t} dt = M'V(\Theta)A,$$

where $V(\Theta)$ is the simple arithmetic mean value of V during the interval t to t-A. Obviously $V(\Theta)A = 1.00$ because the average speed of an express train while in a semaphore block must equal the reciprocal of the time it takes to run the block. Hence $M'V(\Theta)A = M'$. If M' is not constant the proof is a little more complicated. Since the first integral in formula [9] is equal to M' it follows the sum of the second integral and the second bracket must equal zero, *i.e.*, the last named terms must be of opposite sign or both zero.

III

The word « investment » in economic parlance has come to be synonymous with what the accountants would call « additions to investment during the accounting period ». To the accountant the investment of a firm at the close of a period is the total amount of invested capital (less depreciation, etc.) as of that date. When the modern economist speaks of investment in business cycle theory, he usually means the value of additional money invested in capital during some period of time. So conceived, the word investment becomes identical with the phrase « net increase of capital ». And, of course, if the capital is valued at cost less depreciation, which is standard accounting procedure, the net increase of capital must be the same thing as the *purchases* of new (additional) capital goods because a thing cannot acquire a pecuniary cost value unless and until it has been purchased. So one may use the terms, investment, net increase of capital and purchases of new capital more or less interchangeably.

The classical conception of the value of savings in goods identifies it with investment, *i.e.*, the value of savings in goods is defined in such way as to make it exactly equal to the *whole* increase in value of goods capital; just as the common-sense definition of cash savings makes this term exactly equal to the whole increase of cash where there is no borrowing or lending, etc. But the argument has been made in this paper that for analytical reasons it is preferable to define cash savings as only a certain *part* of the increase of cash thereby separating the savings-in-cash concept from new cash income, with which it is merged by the common-sense definition. An analogous argument can be made that it is preferable for economic analysis to identify goods savings not with the whole increase in value of goods capital but only with a part thereof; namely, the part which is not accounted for by new goods income. In that case, goods savings will not equal investment except in the special case where there is no new goods income. No doubt the late Lord Keynes had some dim, intuitive perception of this point of view when he wrote the following in his *Treatise on Money*:

« It might be supposed — and has frequently been supposed — that the amount of investment is necessarily equal to the amount of saving. But reflection will show that this is not the case...

« That saving can occur without any corresponding investment is obvious, if we consider what happens when an individual refrains from spending his money income on consumption... The savers are individually richer by the amount of their savings but the producers who [may] have sold their current output at a lower price than they would have got... are poorer by an equal amount » (5).

The bracketed word « may » was inserted by the present writer because without this word the quotation implies that new goods income must always be insufficient to offset savings, which is not true.

The argument can be put more succinctly and precisely by the use of fairly simple mathematical notation. Let $Y(t)$ represent the total value of all goods capital at time t, valued at cost less depreciation, and let $Z(t)$ represent the rate of purchase of goods capital in the aggregate at the same time. Y will be of the dimension dollars (or lire, etc.) while Z is of the dimension dollars (or lire, etc.) per period of time. Then the variable $K(t)$ may be defined by the following definite integral:

$$Y(t) = \int_{t-K}^{t} Z(t)dt. \qquad [13]$$

(5) J. M. KEYNES, *op. cit.*, I, 172-4. The lamented baron repudiated this point of view in his later works and in his *General Theory* even executed an about-face by stating flatly that savings must always be equal to investment.

From considerations analogous to those involved in the discussion of the meaning of α in formula [3] above, K will be recognized as a constructive average period of turnover of all goods capital. Goods are not homogeneous, like money is, but each separate kind of good may be said to have a period of turnover, say k, and the capital K in formula [13] can be thought of as a weighted average of all the various k's (6).

The variable K is said to be a *constructive* period of turnover because it includes the constructive period of turnover of so-called fixed capital as it is worn out and charged to depreciation. K also includes the constructive period of turnover of such capital as coal which, *in effect*, « turns over » as it is burned up in the creation of other goods. All forms of goods capital have a period of turnover in a constructive sense even if title to them may never change hands. In any case there is a period of elapsed time between the date on which they are acquired by a given firm and the date on which their value has been completely charged off the books and charged into the value of goods sold or to be sold by the firm.

Differentiation of formula [13] will give the value of $Y'(t)$, the instantaneous rate of increase of total goods capital, viz.,

$$Y'(t) = [Z(t) - Z(t-K)] + [K'Z(t-K)] \qquad [14]$$

where K' is the rate of change of the average period of turnover of all goods capital taken with respect to time. K' is of zero dimension, a pure number.

The familiar pair of brackets reappear and serve notice that investment, or the increase in value of goods capital, like the increase of money capital, is brought about by one or both of two quite different operations. In the case of investment these two operations are (a) increasing the rate of purchase of capital, and (b) economizing in the use of existing capital,

(6) It can be shown that K is approximately equal to $\Sigma R_i z_i / \Sigma z_i$, where the z_i are rates of purchase of each kind of good.

which means to make it last longer. The increase of capital effected by the first kind of operation is evaluated by the first bracket in formula [14], while the increase effected by the second kind of operation is evaluated by the second bracket. If a trader holds his rate of purchase of capital constant then the only way he can increase his goods capital is to make his capital last longer, in other words increase the average period of turnover of his capital. This will reduce the rate of outflow of capital and increase the capital owned while input remains constant. The mathematical effect of this kind of action will be registered in the aggregate by the second bracket in formula [14]. On the other hand if a trader keeps his goods capital moving at the same average rate of turnover then the only way he can increase his goods capital is to increase his rate of purchase of goods, and the aggregate economic effect of such activity is registered by the first bracket term in equation [14]. These two methods of increasing capital are largely antithetical and a more trenchant economic analysis is permitted if they are not merged in the single concept of « savings » as is done by the classical systems of thought.

A more sapient analysis of business fluctuations is permitted if the aggregate rate of goods savings is defined as only *that part* of the rate of investement (rate of increase of capital) which is represented by the second bracket in formula [14] just as the aggregate rate of cash savings has been defined by the second bracket in formula [9]. Aggregate savings in goods will then be identified as that part of aggregate net investment which is contributed by those traders who increase their capital by increasing its period of turnover while holding their input of goods constant. The other bracket in formula [14] can be designated suitably as « rate of new goods income ». Thus the aggregate rate of investment will be equal to the total rate of new goods income plus the total rate of goods savings. Investment will be equal to goods savings only in the special case where there is no new goods income.

The fact that the first and second brackets in equation [14] are to some extent antithetical may be demonstrated by expanding the first bracket. Let $Z(t)$ be factored into the product of $Y(t)$ the aggregate value of goods capital, and $R(t)$ its average rate of turnover, where R is *defined* as the quotient of Z divided by Y. R is of the dimension times per period of time. Upon making this substitution and certain adjustments in formula (14) it reads as follows:

$$Y'(t) = \left[\int_{t-K}^{t} Y(t)R(t)dt + \int_{t-K}^{t} R'(t)Y(t)dt\right] + [K'Z(t-K)], \quad [15]$$

where R' is the rate of change of R with respect to time. R' is of the dimension times per period of time per period of time. While R is not in general exactly equal to the reciprocal of K, the average period of turnover of all capital, R will tend to vary inversely with K, and the rates of change of R and K will always be of opposing sign. In fact R and K will behave in such way that the value of the second integral in the first bracket of formula [15] will always be numerically equal to and of opposite sign to the value of the second bracket (7). Thus a positive savings in goods by the proposed new definition will always have a negative effect on new goods income, *i.e.*, positive savings will always *tend* to cause a reduction in aggregate goods income. It was this sort of thing which Lord Keynes apparently had in mind when he noted the paradoxical nature of saving from the social point of view in the passage quoted above. But savings will not *necessarily* cause new goods income to be negative because of the presence of the *first* definite integral in the complete expression for new goods income. If the value of this integral

(7) The easiest and perhaps the only way to prove this statement is true is to first prove that the first integral in equation [15] is exactly equal to Y', the variable on the left-hand side of the equation. This being so all the terms taken together on the right-hand side which are not included in the first integral must sum to zero. Since there are only two more terms, namely the second integral and the second bracket, they must be equal and of opposite sign. Proof that

$$Y'(t) = \int_{t-K}^{t} Y'(t)R(t)dt$$

would follow along the lines indicated in proof of the analogous theorem with respect to increase of money as given in footnote (4) above.

is positive and sufficiently large it can more than offset any reduction of goods income caused by goods savings. This first integral in the first bracket obviously evaluates the rate of purchase of new goods capital. It is equal at all times to the rate of increase of capital or rate of investment. This integral imparts a constructive rate of turnover to the new capital by multiplying Y' by R, which multiplication implies that additions to capital are in general turning over at the same rate as « old » capital. Then this multiplying effect is exactly canceled, dimensionally and numerically, by integrating the product over the interval K.

If a hermit valued his capital at say, cost in labor hours, he might have a positive value for Y', his rate of increase of capital; but he would have no rate of purchase of new capital because R would be zero for him. For him the first integral in formula [15] would be meaningless and so would the whole equation. But in an exchange economy, nearly all capital is in motion between traders and the first integral in the equation measures the rate of exchange of new capital over an interval of time just sufficient to make it equal to the rate of increase of capital. If one were to integrate the expression $Y'(t)R(t)$ over an interval of time *other* than the period K, the resulting definite integral would still evaluate that part of the increase in rate of goods income over the given interval which is produced by the use of new capital, but it would no longer be equal to the rate of increase of capital.

The second integral in formula [15] evaluates that part of the increase in rate of goods income over the interval K which is caused by variation in rate of utilization of existing capital. As pointed out before this part will be negative when goods savings are positive, and this part always cancels out exactly the effect of goods savings upon investment *except* in so far as the goods savings may have a positive effect upon the first integral in the first bracket of formula [15] by virtue of economic considerations. There is no reason from the purely mathematical point of view why traders should not increase their capital by the exact amount of their savings in goods as defined in this paper, in spite of the presence of the inverse of savings in equation [15]. Thus, equation [15] above might conceivably have the following statistical evaluation (8):

Investment	= Purchases of New Capital
+ $ 20 Billion	+ $ 20 Billion
+ Inverse Savings	+ Savings
− $ 20 Billion	+ $ 20 Billion

In that case new goods income will be zero and the entire investment would be achieved by savings operations. It is quite possible to increase capital by slowing down its rate of utilization while holding input of goods constant, and many individual firms do accomplish an increase of capital in this manner.

IV

But *in the aggregate* the first integral in equation [15] and the second bracket appear to have opposite signs most of the time, because of psychological considerations and the paradoxical effect of savings as noted by Lord Keynes. In short, aggregate goods savings are generally negative when investment is positive and vice-versa. When investment is positive there generally exists an atmosphere of business confidence and optimism, and under this stimulus people tend to turn capital over in a shorter rather than a longer time, which means savings become negative rather than positive.

The proposed new definition of goods savings will impose additional burdens on the statistician and upon the popular understanding; but it has a tremendous advantage for econo-

(8) In strict logic Y' in formula [15] is not the investment of a given period, but the rate of investment at a given moment. The actual investment of a finite interval ε, ending at time $t = \tau$ would be given by taking the definite integral of Y' over that interval. Similarly, the savings of an interval of time equal to ε would be given by the following double integral:

$$-\int_{\tau-\varepsilon}^{\tau} \int_{t-\varepsilon}^{t} R'(t)\, Y(t)dtdt.$$

Compare this writer's *La Dimensione del Tempo nella Misura del Risparmio*, nel « Giornale degli Economisti e Annali di Economia », gennaio-febbraio, 1951.

mic analysis; namely, that it calls sharp attention to the cardinal paradox of modern economic life under a free-enterprise system. This cardinal paradox is as follows: that while an individual can, and often does increase his capital by conserving its rate of utilization, in the aggregate a nation does not expand its wealth so much by this method as it does by employing existing capital freely in the exploitation of new resources and the creation of new income. The business, financial and political leaders of the free nations must come to grasp and understand the egregious difference between individual savings and savings in the aggregate if the free enterprise system is to survive.